

TORSION ORDERS OF COMPLETE INTERSECTIONS

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ABSTRACT. By a classical method due to Roitman, a complete intersection X of sufficiently small degree admits a rational decomposition of the diagonal. This means that some multiple of the diagonal by a positive integer N , when viewed as a cycle in the Chow group, has support in $X \times D \cup F \times X$, for some divisor D and a finite set of closed points F . The minimal such N is called the torsion order. We study lower bounds for the torsion order following the specialization method of Voisin, Colliot-Thélène and Pirutka. We give a lower bound for the generic complete intersection with and without point. Moreover, we use methods of Kollár and Totaro to exhibit lower bounds for the very general complete intersection.

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INTRODUCTION

Decomposition of the diagonal has played a prominent role in recent progress on stable rationality questions. For a rationally connected variety over a field k , there is a minimal integer $\mathrm{Tor}_k(X) \geq 1$ such that the multiple of the diagonal $\mathrm{Tor}_k(X) \cdot \Delta_X$, when viewed in the Chow group of $X \times X$, is supported in $X \times D \cup F \times X$, for some divisor D and some finite set of closed points F . We will call $\mathrm{Tor}_k(X)$ the *torsion order* of X ; it is a stable birational invariant which equals 1 if X is stably rational and in general gives

an upper bound on the exponent of the unramified cohomology of X . This invariant is also studied by Kahn [17]. In a proper flat family the torsion order of a fiber divides the torsion order of the generic fiber (see Lemma 1.5 for the precise statement). One can thus deduce a non-trivial torsion order from a non-trivial torsion order of a cleverly chosen degeneration.

This method was pioneered by Voisin [29]. It was significantly simplified and applied by Colliot-Thélène and Pirutka to show the non-rationality of a very general quartic fourfold [6] by using a degeneration to a classical example of Artin and Mumford (after a “universally CH_0 -trivial” resolution of singularities [6, Definitions 1.1, 1.2]), which is a unirational but non-rational variety. The non-trivial 2-torsion in its Brauer group forces non-triviality of the torsion order (in fact, it implies that the torsion order is even). Totaro [28] used Voisin’s method combined with work of Kollár [18] to improve Kollár’s non-rationality results for hypersurfaces in *loc. cit.* Roughly speaking, Totaro showed how, for large enough degree, a general hypersurface of even degree degenerates to an inseparable degree 2 cover in characteristic 2 whose resolution of singularities can be shown to support non-vanishing differential forms. As for the Brauer group, action of correspondences (and the fact that the singularities of the degeneration are “not too bad”) shows divisibility of the torsion order by 2.

In this paper we study the torsion order of complete intersections in projective space. A classical result by Roitman, which we recall in Proposition 4.1, establishes an upper bound stating that a complete intersection X of multi-degree (d_1, \dots, d_r) in \mathbb{P}_k^{n+r} (over any field k) with $\sum_{i=1}^r d_i \leq n+r$ satisfies $\mathrm{Tor}_k(X) \mid \prod_{i=1}^r (d_i!)$. Our first result is a lower bound for a generic complete intersection.

Theorem (Theorem 5.5, Corollary 5.6). *Let $\mathcal{Y} := \prod_{i=1}^r \mathbb{P}(H^0(\mathbb{P}_k^{n+r}, \mathcal{O}(d_i))^\vee)$, and let $\mathcal{X} \subset \mathcal{Y} \times \mathbb{P}_k^{n+r}$ be the incidence variety*

$$\mathcal{X} = \{(f_1, \dots, f_r, x) \in \mathcal{Y} \times \mathbb{P}_k^{n+r} \mid f_1(x) = \dots = f_r(x) = 0\}.$$

We denote by K the quotient field of \mathcal{Y} , and let X/K be the generic fiber of the family $\mathcal{X} \rightarrow \mathcal{Y}$. For an integer $d \geq 1$, let $d!^$ be the least common multiple of the integers $1, \dots, d$. The following holds:*

- i) $\mathrm{Tor}_K(X)$ is divisible by $\prod_{i=1}^r d_i!^*$,*
- ii) $\mathrm{Tor}_{K(X)}(X \otimes_K K(X))$ is divisible by $\frac{\prod_{i=1}^r d_i!^*}{d_1 \cdots d_r}$.*

The invariant which detects divisors of the torsion order in the first part of theorem is the index of a variety, that is, the image of the Chow group of zero cycles via the degree map. The index of X/K is given by $d_1 \cdots d_r$. Divisibility of the torsion order by other integers of the form $i_1 \cdots i_r$ with $1 \leq i_j \leq d_j$ is shown by degeneration to a union of complete intersections with lower degrees and using induction.

We also consider the generic cubic hypersurface with a line, and use Theorem 5.5 to show that this has torsion order exactly 2 (Example 5.8). We show the existence of a cubic threefold over $K = \mathbb{Q}_p((x))$ or $K = \mathbb{F}_p((t))((x))$,

having a K -point and torsion order divisible by 2 (Example 5.9); more generally, we construct examples of cubic hypersurfaces of dimension n over a field $K = k((x))$, where k is a field of characteristic zero and u -invariant at least $n + 1$, which have a K -point and for which 2 divides the torsion order. This last series of examples is taken over from [8], with the kind permission of the author, and it gives an improvement over a construction in an earlier version of this paper, which relied on Rost's degree formula. We should mention that other examples of this kind already exist in the literature, see for example [6, Théorème 1.21], where cubic threefolds over a p -adic field with non-zero torsion order are constructed, as well as examples over $\mathbb{F}_p((x))$ [6, Remarque 1.23]; both examples have a rational point.

Our second result concerns the torsion order of very general complete intersections over algebraically closed fields of characteristic zero. The idea of the proof is as in the papers of Kollár and Totaro. We are able to generalize the results on the Hodge cohomology of the degeneration in characteristic p to Hodge–Witt cohomology. In this way we can establish results on divisibility by powers of p .

Theorem (Theorem 7.2). *Let k be an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}_k^{n+r}$ be a very general complete intersection of multi-degree d_1, d_2, \dots, d_r such that $d' := \sum_{i=1}^r d_i \leq n + r$ and $n \geq 3$. Let p be a prime, $m \geq 1$, and suppose*

$$d_i \geq p^m \cdot \left\lceil \frac{n + r + 1 - d' + d_i}{p^m + 1} \right\rceil$$

for some i , where $\lceil \cdot \rceil$ denotes the ceiling function. Moreover, we assume that p is odd or n is even. Then $p^m | \text{Tor}_k(X)$.

For example, if $\sum_{i=1}^r d_i = n + r$ and $n \geq 3$, which is the extreme case, then $d_i | \text{Tor}_k(X)$ if d_i is odd or n is even. For hypersurfaces and $m = 1$, the theorem is due to Totaro, and we give a short proof of the straight-forward generalization to complete intersections and the case $m = 1$ in Theorem 6.1. We should mention that our Theorem 6.1 and Theorem 7.2 are actually a bit stronger, in that we prove the same divisibility result for the torsion orders of level $n - 2$ (see below), which automatically divide the torsion orders described above.

The paper is divided into seven sections. Section 1 contains the definition and basic properties of the torsion order. Following a suggestion of Claire Voisin, we consider decompositions of the diagonal of higher “niveau level” and the associated torsion invariants; we also describe some elementary specialization results. In section 2 we recall from Colliot-Thélène and Pirutka the notion of a universally CH_0 -trivial morphism and a related notion, that of a totally CH_0 -trivial morphism. Behavior under a combination of degeneration and modification by a birational totally CH_0 -trivial morphism, which is the basic tool used for divisibility results, is the focus of section 3; in this section we follow [6] and extend their specialization results to cover

decompositions of higher level. We recall Roitman's theorem in section 4 and discuss the case of the generic complete intersection in section 5. We recall Totaro's arguments leading to the divisibility results for the torsion order of a very general complete intersection in section 6 and conclude by proving our refined version in section 7.

We would like to thank the referees very much for thoroughly reading the paper and suggesting improvements. We are especially grateful to the referee who suggested the statement and proof of Lemma 7.1. This result enabled us to improve an earlier version of our Theorem 7.2 to the statement on higher torsion orders mentioned above. We are also grateful to Jean-Louis Colliot-Thélène, who very kindly allowed us to include some of the results of his paper [8]. This led to a new result (Lemma 1.8) on specialization of decompositions of the diagonal, derived from [8, Lemma 2.2], and Example 5.9 mentioned above, a version of which appears as [8, Théorème 2.4].

1. TORSION ORDERS

Let k be a field and X a k -scheme of finite type. If A is a presheaf on X_{Zar} , we let

$$A(X(i)) := \text{colim}_F A(X \setminus F)$$

where F runs over all closed subsets of X with $\dim_k F \leq i$. We extend this notation to products, defining for a presheaf A on $(X \times_k Y)_{\text{Zar}}$

$$A(X(i) \times Y(j)) = \text{colim}_{F,G} A((X \setminus F) \times (Y \setminus G)).$$

For example, the contravariant functoriality of the classical Chow groups for open immersions allows us to apply this notation to $A(X) := \text{CH}_n(X)$ for some n .

Let k be a field with algebraic closure \bar{k} . We say that a finite type k -scheme X is *generically reduced* if X is reduced at each generic point. We call a reduced finite type k -scheme X *separable* over k if the total quotient ring $k(X)$ is a product of separably generated field extensions of k . For X an arbitrary finite type k -scheme, call X separable over k if X_{red} is so. We note that for X generically reduced and separable over k , $X \times_k \bar{k}$ is also generically reduced. A closed subset D of a finite type k -scheme X is called *nowhere dense* if D contains no generic point of X .

Definition 1.1. Let k be a field and let X be a reduced proper k -scheme of pure dimension d over k .

1. For $i = 0, 1, 2, \dots$, the i th torsion order of X , $\text{Tor}_k^{(i)}(X) \in \mathbb{N}_+ \cup \{\infty\}$, is the order of the image of the diagonal $\Delta_X \subset X \times_k X$ in $\text{CH}_d(X(i) \times X(d-1))$. We write $\text{Tor}_k(X)$ for $\text{Tor}_k^{(0)}(X)$ and call this the torsion order of X .
2. Suppose X is separable over k . For $1 \leq i < j \leq 3$, let $p_{ij} : X \times_k X \times_k X \rightarrow X \times_k X$ denote the projection on the i th and j th factors, and let $\Delta_{ij} \subset X \times_k X \times_k X$ denote the pullback $p_{ij}^{-1}(\Delta_X)$. Consider the Cartesian

diagram

$$\begin{array}{ccc} X_{k(X \times X)} & \xrightarrow{\tilde{j}} & X \times_k X \times_k X \\ \downarrow & & \downarrow p_{23} \\ \mathrm{Spec} k(X \times X) & \xrightarrow{i} & X \times_k X. \end{array}$$

Let $\eta_1 - \eta_2 \in \mathrm{CH}_0(X_{k(X \times_k X)})$ denote the class of the pullback $\tilde{j}^*(\Delta_{12} - \Delta_{13})$. The *generic torsion order* of X , $g\mathrm{Tor}_k(X) \in \mathbb{N}_+ \cup \{\infty\}$, is the order of $\eta_1 - \eta_2$ in $\mathrm{CH}_0(X_{k(X \times X)})$.

3. We say that X admits a decomposition of the diagonal of order N and level i if there is a nowhere dense closed subset D , a closed subset Z of X with $\dim_k Z \leq i$ and cycles γ, γ' on $X \times X$, with γ supported in $X \times D$, γ' supported in $Z \times X$ and with

$$N \cdot [\Delta_X] = \gamma' + \gamma$$

in $\mathrm{CH}_d(X \times_k X)$.

4. Suppose X is geometrically integral. For an integer $N \geq 1$, we say that X admits a decomposition of the diagonal of order N if there is a 0-cycle x on X , a proper closed subset D of X and a dimension d cycle γ on $X \times_k X$, supported in $X \times D$, such that

$$N \cdot [\Delta_X] = x \times X + \gamma$$

in $\mathrm{CH}^d(X \times_k X)$. We say that X admits a \mathbb{Q} -decomposition of the diagonal if X admits a decomposition of order N for some N , and that X admits a \mathbb{Z} -decomposition of the diagonal if X admits a decomposition of the diagonal of order 1.

5. Let $\deg : \mathrm{CH}_0(X) \rightarrow \mathbb{Z}$ be the degree map. For X smooth and integral, the *index* of X is the positive generator I_X of the subgroup $\deg \mathrm{CH}_0(X) \subset \mathbb{Z}$. Equivalently, I_X is the g.c.d. of all degrees $[k(x) : k]$ as x runs over closed points of X . We extend the definition of the index to proper, integral, separable k -schemes Y by defining I_Y to be the g.c.d. of all degrees $[k(y) : k]$ as y runs over closed points of the smooth locus Y_{sm} of Y (which is dense in Y , as Y is separable over k).

Remarks 1.2. 1. Suppose X has pure dimension d over k and is geometrically integral. Since the only dimension d cycles $\gamma_{(0)}$ on $X \times X$, supported on $Z_{(0)} \times X$ with $Z_{(0)} \subset X$ a dimension zero closed subset are of the form $\gamma_{(0)} = x \times X$ for some 0-cycle x on X , a decomposition of the diagonal of order N and level 0 is the same as decomposition of the diagonal of order N .

2. We extend the definition of $\mathrm{Tor}_k^{(i)}(X)$ to all proper, equi-dimensional k -schemes by setting $\mathrm{Tor}_k^{(i)}(X) := \mathrm{Tor}_k^{(i)}(X_{red})$.

3. We will often use an equivalent formulation of Definition 1.1(3), namely, that X admits a decomposition of the diagonal of order N and level i if

there is a closed subset D containing no generic point of X and a closed subset Z of X with $\dim_k Z \leq i$ such that

$$N \cdot j^*[\Delta_X] = 0$$

in $\mathrm{CH}_d((X \setminus Z) \times_k (X \setminus D))$, where $j : (X \setminus Z) \times_k (X \setminus D) \rightarrow X \times_k X$ is the inclusion. This equivalence follows from the localization sequence

$$\mathrm{CH}_d(Z \times_k X \cup X \times_k D) \xrightarrow{i_*} \mathrm{CH}_d(X \times_k X) \xrightarrow{j^*} \mathrm{CH}_d((X \setminus Z) \times_k (X \setminus D)) \rightarrow 0$$

and the surjection

$$\mathrm{CH}_d(Z \times_k X) \oplus \mathrm{CH}_d(X \times_k D) \rightarrow \mathrm{CH}_d(Z \times_k X \cup X \times_k D).$$

4. Decompositions of the diagonal for *smooth* proper k -varieties have been considered in [2, 6, 28] and by many others. Here we have extended the definition to proper, equi-dimensional, but not necessarily smooth k -schemes.

Lemma 1.3. *Let X be a proper k -scheme of pure dimension d over k .*

1. *If $\mathrm{Tor}_k^{(i)}(X)$ is finite then so $\mathrm{Tor}_k^{(i+1)}(X)$ and in this case, $\mathrm{Tor}_k^{(i+1)}(X)$ divides $\mathrm{Tor}_k^{(i)}(X)$.*

2. *X admits a decomposition of the diagonal of order N and level i if and only if $\mathrm{Tor}_k^{(i)}(X)$ divides N ; if X is geometrically integral, then X admits a decomposition of the diagonal of order N if and only if $\mathrm{Tor}_k(X)$ divides N and X does not admit a \mathbb{Q} -decomposition of the diagonal if and only if $\mathrm{Tor}_k(X) = \infty$.*

3. *Suppose X is smooth over k and geometrically integral. If $\mathrm{Tor}_k(X)$ is finite then so is $g\mathrm{Tor}_k(X)$ and $g\mathrm{Tor}_k(X)$ divides $\mathrm{Tor}_k(X)$.*

4. *Suppose X is separable over k and let $L \supset k$ be a field extension. If $\mathrm{Tor}_k^{(i)}(X)$ is finite, then so is $\mathrm{Tor}_L^{(i)}(X_L)$ and in this case, $\mathrm{Tor}_L^{(i)}(X_L)$ divides $\mathrm{Tor}_k^{(i)}(X)$. If L is finite over k then $\mathrm{Tor}_k^{(i)}(X)$ is finite if and only if $\mathrm{Tor}_L^{(i)}(X_L)$ is finite and in this case $\mathrm{Tor}_k^{(i)}(X)$ divides $[L : k] \cdot \mathrm{Tor}_L^{(i)}(X_L)$. The corresponding statements hold replacing $\mathrm{Tor}^{(i)}$ with $g\mathrm{Tor}$.*

5. *X admits a decomposition of the diagonal of level i and order N if and only if there is a closed subset $Z \subset X$ of dimension $\leq i$ such that the pull-back of Δ_X to $(X \setminus Z) \times_k \mathrm{Spec} k(X)$ via the inclusion*

$$(X \setminus Z) \times_k \mathrm{Spec} k(X) \rightarrow X \times_k X$$

has order dividing N in $\mathrm{CH}_0((X \setminus Z) \times_k \mathrm{Spec} k(X))$.

Proof. (1) follows from the existence of the restriction homomorphism

$$\mathrm{CH}_d((X \setminus F) \times_k (X \setminus D)) \rightarrow \mathrm{CH}_d((X \setminus F') \times_k (X \setminus D))$$

for $F \subset F'$. (2) follows from the localization sequence for $\mathrm{CH}_*(-)$, as in Remark 1.2(3).

For (3), suppose

$$N \cdot [\Delta_X] = x \times X + \gamma$$

in $\mathrm{CH}_d(X \times_k X)$ for x and γ as in Definition 1.1. Since X is smooth and proper, we have for every field extension F of k , the action of $\mathrm{CH}_d(X_F \times_F$

X_F) on $\mathrm{CH}_n(X_F)$ as correspondences (see [11]), that is, for $\alpha \in \mathrm{CH}_d(X_F \times_F X_F)$ and $\rho \in \mathrm{CH}_n(X_F)$, one has the well-defined element

$$\alpha^*(\rho) := p_{1*}(p_2^*\rho \cdot \alpha).$$

Acting by the correspondence $N \cdot \Delta_{X_{k(X \times_k X)}}^*$ on $\mathrm{CH}_0(X_{k(X \times_k X)})$ gives

$$N \cdot (\eta_1 - \eta_2) = x - x = 0$$

and thus $g\mathrm{Tor}_k(X)$ divides N . Applying (2) gives (3).

For (4), the first assertion follows by applying the pull-back in CH_d for $X_L \times_L X_L \rightarrow X \times_k X$ and using (2). The second part follows by applying the pushforward map $\mathrm{CH}_d(X_L \times_L X_L) \rightarrow \mathrm{CH}_d(X \times_k X)$ and using (2), and the assertion for $g\mathrm{Tor}_k(X)$ follows similarly by applying the pushforward map $\mathrm{CH}_d(X_{L(X \times_k X)}) \rightarrow \mathrm{CH}_d(X_{k(X \times_k X)})$.

The last assertion (5) follows from the identity

$$\mathrm{CH}_0((X \setminus Z) \times_k \mathrm{Spec} k(X)) = \varinjlim_{D \subset X} \mathrm{CH}_d((X \setminus Z) \times_k (X \setminus D))$$

where the limit is over all closed $D \subset X$ containing no generic point of X . \square

Remark 1.4. We have restricted our attention to proper k -schemes for the definitions of torsion orders and decompositions of the diagonal. Even though the definitions would make sense for non-proper equi-dimensional k -schemes, a naive extension is probably not useful. Possibly replacing Chow groups with Suslin homology would make more sense: following Lemma 1.3, one could define $\mathrm{Tor}^{(i)}(X)$ for an equi-dimensional finite type k -scheme as the order of the restriction of Δ_X to $X \times_k \mathrm{Spec} k(X)$ in the quotient group

$$\varinjlim_{Z \subset X} H_0^{\mathrm{Sus}}(X \times_k \mathrm{Spec} k(X)) / \mathrm{im}(H_0^{\mathrm{Sus}}(Z \times_k \mathrm{Spec} k(X)))$$

where $Z \subset X$ runs over all closed subsets of dimension at most i . We will not investigate properties of these torsion orders for non-proper k -schemes here.

Here is the first in a series of elementary but useful specialization lemmas.

Lemma 1.5. *Let \mathcal{O} be a noetherian regular local ring $f : \mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}$ a proper flat morphism, with \mathcal{X} equi-dimensional over $\mathrm{Spec} \mathcal{O}$ of relative dimension d , $X \rightarrow \mathrm{Spec} K$ the generic fiber, $Y \rightarrow \mathrm{Spec} k$ the special fiber. We suppose that, for each $z \in \mathrm{Spec} \mathcal{O}$, the fiber \mathcal{X}_z is generically reduced and separable over $k(z)$. Fix an integer i .*

1. *If $\mathrm{Tor}_K^{(i)}(X)$ is finite, then so is $\mathrm{Tor}_k^{(i)}(Y)$, and $\mathrm{Tor}_k^{(i)}(Y)$ divides $\mathrm{Tor}_K^{(i)}(X)$.*
2. *If $g\mathrm{Tor}_K(X)$ is finite, then so is $g\mathrm{Tor}_k(Y)$, and $g\mathrm{Tor}_k(Y)$ divides $g\mathrm{Tor}_K(X)$.*
3. *Let \bar{k} and \bar{K} be the respective algebraic closures of k and K , and suppose either K has characteristic zero, or that \mathcal{O} is excellent. If $\mathrm{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}})$ is finite, then so is $\mathrm{Tor}_{\bar{k}}^{(i)}(Y_{\bar{k}})$, and $\mathrm{Tor}_{\bar{k}}^{(i)}(Y_{\bar{k}})$ divides $\mathrm{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}})$.*

Proof. We use the definition of $\mathrm{CH}_d(X(i) \times X(d-1))$ as a limit to reduce to making computations in groups of the form $\mathrm{CH}_d((X \setminus Z) \times (X \setminus D))$ where Z, D are closed subsets of X with $\dim Z \leq i$, $\dim D \leq d-1$. We may stratify $\mathrm{Spec} \mathcal{O}$ by regular closed subschemes $Z_0 \subset \dots \subset Z_r = \mathrm{Spec} \mathcal{O}$, with Z_i of Krull dimension i . This gives us the DVRs $\mathcal{O}_i := \mathcal{O}_{Z_i, Z_{i-1}}$ and the restriction of \mathcal{X} to $\mathcal{X}_i \rightarrow \mathrm{Spec} \mathcal{O}_i$. Regarding the proof of (3), if the original local ring \mathcal{O} has characteristic zero quotient field, we may stratify $\mathrm{Spec} \mathcal{O}$ as above so that each DVR \mathcal{O}_i has characteristic zero quotient field, and if \mathcal{O} is excellent, so are each of the \mathcal{O}_i . Proving the result for each of the families \mathcal{X}_i gives the result for \mathcal{X} , which reduces us to the case of a DVR \mathcal{O} .

In this case, suppose we have a relation

$$(1.1) \quad N \cdot \Delta_X = 0$$

in $\mathrm{CH}_d((X \setminus Z) \times (X \setminus D))$, with $\dim_K Z \leq i$ and D nowhere dense. Taking the closures \bar{Z} and \bar{D} in \mathcal{X} , and letting $Z_0 = Y \cap \bar{Z}$, $D_0 = Y \cap \bar{D}$, we have the specialization homomorphism (see for example [11, 6.3.7])

$$sp : \mathrm{CH}_d((X \setminus Z) \times_K (X \setminus D)) \rightarrow \mathrm{CH}_d((Y \setminus Z_0) \times_k (Y \setminus D_0))$$

associated to the family

$$\mathcal{X} \times_{\mathcal{O}} \mathcal{X} \setminus \bar{Z} \times \mathcal{X} \cup \mathcal{X} \times \bar{D} \rightarrow \mathrm{Spec} \mathcal{O}.$$

Note that, as \mathcal{O} is a DVR, the closure \bar{Z} is equi-dimensional over $\mathrm{Spec} \mathcal{O}$, and thus $\dim_k Z_0 \leq i$; similarly, D_0 is nowhere dense in Y . Since $\mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}$ is flat and the fibers are generically reduced, we have

$$sp(\Delta_X) = \Delta_Y$$

in $\mathrm{CH}_d((Y \setminus Z_0) \times_k (Y \setminus D_0))$, so applying sp to (1.1) proves (1).

The proof of (2) is a similar specialization argument. Indeed, we reduce as before to the case of a DVR \mathcal{O} . Due to the generic separability assumption, there is a dense open subscheme \mathcal{U} of $\mathcal{X} \times_{\mathcal{O}} \mathcal{X}$ that is smooth over $\mathrm{Spec} \mathcal{O}$, with special fiber dense in $Y \times_k Y$. If now τ is a generic point of $Y \times_k Y$, let \mathcal{R} be the local ring $\mathcal{O}_{\mathcal{U}, \tau}$. Then \mathcal{R} is a DVR and we may consider the \mathcal{R} -scheme $\mathcal{X} \otimes_{\mathcal{O}} \mathcal{R} \rightarrow \mathrm{Spec} \mathcal{R}$. The quotient field F of \mathcal{R} is one of the field factors of $k(X \times_K X)$ and the residue field \mathfrak{f} of \mathcal{R} is the factor of $k(Y \times_k Y)$ corresponding to τ . Let $\eta_i^X, \eta_i^Y, i = 1, 2$ denote the images of the “generic” points used to define $g\mathrm{Tor}_K(X)$, resp. $g\mathrm{Tor}_k(Y)$ in $\mathrm{CH}_0(\mathcal{X}_F)$, resp. $\mathrm{CH}_0(Y_{\mathfrak{f}})$. Applying the specialization homomorphism

$$sp : \mathrm{CH}_0(\mathcal{X}_F) \rightarrow \mathrm{CH}_0(Y_{\mathfrak{f}})$$

to a relation $N \cdot (\eta_1^X - \eta_2^X)$ in $\mathrm{CH}_0(\mathcal{X}_F)$ shows that $N \cdot (\eta_1^Y - \eta_2^Y) = 0$ in $\mathrm{CH}_0(Y_{\mathfrak{f}})$ for each generic point τ , and thus $g\mathrm{Tor}_k(Y)$ divides N .

For (3), we note that there is a finite extension L of K so that

$$\mathrm{Tor}_K^{(i)}(X_K) = \mathrm{Tor}_L^{(i)}(X_L) = \mathrm{Tor}_F^{(i)}(X_F)$$

for all finite extensions F of L . Since either K has characteristic zero or \mathcal{O} is excellent, the normalization \mathcal{O}^N of \mathcal{O} in L is a semi-local principle ideal

ring, finite over \mathcal{O} (the characteristic zero case follows from [27, Chap. V, Thm. 7] and the excellent case follows from [21, Theorem 78]). Thus, after replacing \mathcal{O} with the localization \mathcal{O}' of \mathcal{O}^N at a maximal ideal, and replacing \mathcal{X} with $\mathcal{X}' := \mathcal{X} \otimes_{\mathcal{O}} \mathcal{O}'$, we may assume that $\mathrm{Tor}_K^{(i)}(X) = \mathrm{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}})$. Since $\mathrm{Tor}_{\bar{k}}^{(i)}(Y_{\bar{k}})$ divides $\mathrm{Tor}_k^{(i)}(Y)$ by Lemma 1.3(4), (3) follows from (1). \square

A global version of Lemma 1.5(3) follows by an argument using Hilbert schemes and Chow varieties. See [29, Theorem 1.1 and Prop. 1.4] or [6, Appendix B] for similar statements.

Corollary 1.6. *Let $p : \mathcal{X} \rightarrow B$ be a flat, equi-dimensional and projective family over a scheme B of finite type over a field k and let b_0 be a point of B . We suppose that each geometric fiber of p is generically reduced. Fix an integer $i \geq 0$. Then there is a countable union of closed subsets $F = \bigcup_{i=1}^{\infty} F_i$ with $b_0 \notin F$ such that for all $b \in B \setminus F$, the geometric fiber $\mathcal{X}_{\overline{k(b)}}$ satisfies $\mathrm{Tor}^{(i)}(\mathcal{X}_{\overline{k(b_0)}}) \mid \mathrm{Tor}^{(i)}(\mathcal{X}_{\overline{k(b)}})$. Here we use the convention that $N \mid \infty$ for all $N \in \mathbb{N}_+ \cup \{\infty\}$ and $\infty \mid N \Rightarrow N = \infty$.*

Proof. Let d be the relative dimension of \mathcal{X} over B . For a positive integer M , let $\mathcal{S}(M)$ be the set of $b \in B$ such that M does not divide $\mathrm{Tor}^{(i)}(\mathcal{X}_{\overline{k(b)}})$. Taking $M = \mathrm{Tor}^{(i)}(\mathcal{X}_{\overline{k(b_0)}})$ and $F = \mathcal{S}(M)$, it suffices to show that $\mathcal{S}(M)$ is a countable union of closed subsets of B .

We first show that $\mathcal{S}(M)$ is closed under specialization. Indeed, if we have a specialization $b \rightsquigarrow \bar{b}$ with $b \in \mathcal{S}(M)$, then there is an excellent DVR \mathcal{O} and a morphism $\mathrm{Spec} \mathcal{O} \rightarrow B$ with b the image of the generic point of $\mathrm{Spec} \mathcal{O}$ and \bar{b} the image of the closed point. Indeed, let C be the closure of b in B , blow-up $\mathrm{Spec} \mathcal{O}_{C,\bar{b}}$ along \bar{b} , normalize to obtain a normal scheme $\pi : T \rightarrow \mathrm{Spec} \mathcal{O}_{C,\bar{b}}$ of finite type over $\mathcal{O}_{C,\bar{b}}$, choose a generic point t of the Cartier divisor $\pi^{-1}(\bar{b})$ on T and take $\mathcal{O} := \mathcal{O}_{T,t}$. The local ring $\mathcal{O}_{C,\bar{b}}$ is excellent since C is of finite type over a field, and the operations used in constructing \mathcal{O} from $\mathcal{O}_{C,\bar{b}}$ all preserve excellence (see [21, Chapters 12, 13]). Pulling back \mathcal{X} to $\mathrm{Spec} \mathcal{O}$, it follows from Lemma 1.5 and Lemma 1.10(3) that \bar{b} is also in $\mathcal{S}(M)$.

Since $\mathcal{S}(M)$ is closed under specialization, it suffices to show that, for each affine open subscheme U of B , $\mathcal{S}(M) \cap U$ is a countable union of closed subsets of U . Thus, we may assume that B is affine, and that \mathcal{X} is a closed subscheme of $B \times \mathbb{P}_k^n$ for some n , with $p : \mathcal{X} \rightarrow B$ the restriction of the projection.

By standard Hilbert scheme arguments, there is a projective B -scheme $q : \mathcal{Y} \rightarrow B$ such that the geometric points of \mathcal{Y} consists of triples (b, Z, D) , with b a geometric point of B , $Z \subset \mathcal{X}_{\overline{k(b)}}$ a closed subscheme of dimension $j \leq i$ and $D \subset \mathcal{X}_{\overline{k(b)}}$ a closed subscheme of dimension $< d$, and with Z and D having fixed Hilbert polynomials (chosen in advance). Similarly, using Chow varieties, there is a projective B -scheme $r : \mathcal{W} \rightarrow B$ whose geometric

points consists of triples (b, W^+, W^-) with $W^+, W^- \subset \mathcal{X}_{\overline{k(b)}} \times_{\overline{k(b)}} \mathcal{X}_{\overline{k(b)}} \times \mathbb{P}^1$ dimension $d+1$ effective cycles of some fixed bi-degrees (chosen in advance). \mathcal{W} contains the open subscheme \mathcal{W}^0 of triples (b, W^+, W^-) such that both W^+ and W^- have no component contained in $\mathcal{X}_{\overline{k(b)}} \times_{\overline{k(b)}} \mathcal{X}_{\overline{k(b)}} \times \{0, \infty\}$.

Fix an integer $N > 0$. In $\mathcal{Y} \times_B \mathcal{W}^0$ we have the closed subscheme \mathcal{R}_N whose geometric points consists of tuples (b, Z, D, W^+, W^-) such that the cycle

$$(\mathcal{X}_{\overline{k(b)}} \times_{\overline{k(b)}} \mathcal{X}_{\overline{k(b)}} \times 0) \cdot (W^+ - W^-)$$

is supported in $Z \times \mathcal{X}_{\overline{k(b)}} \times 0 \cup \mathcal{X}_{\overline{k(b)}} \times D \times 0$, and

$$(\mathcal{X}_{\overline{k(b)}} \times_{\overline{k(b)}} \mathcal{X}_{\overline{k(b)}} \times \infty) \cdot (W^+ - W^-) = N \cdot \Delta_X \times \infty.$$

The image of \mathcal{R}_N under the projection $\mathcal{R}_N \rightarrow B$ is a constructible subset of B . We vary the choice of N over integers not divisible by M , and also vary over all choices of Hilbert polynomials (for dimension $\leq i$ closed subschemes Z and closed subschemes D of dimension $< d$) and all bi-degrees for the effective cycles W^+, W^- . As this set of choices is countable, it follows that $\mathcal{S}(M)$ is a countable union of constructible subsets of B . As $\mathcal{S}(M)$ is closed under specialization, the proof is complete. \square

Next, we prove a modification of the specialization Lemma 1.5. A related result may be found in [28, Lemma 2.4].

Lemma 1.7. *Let \mathcal{O} be a discrete valuation ring with quotient field K and residue field k . Let $f : \mathcal{X} \rightarrow \text{Spec } \mathcal{O}$ be a flat and proper morphism of dimension d over $\text{Spec } \mathcal{O}$ with generic fiber X and special fiber Y . We suppose Y is a union of closed subschemes, $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 having no common components, and that X and $Y_1 \setminus Y_2$ are generically reduced. Suppose in addition that X admits a decomposition of the diagonal of order N and level i . Then there is an identity in $\text{CH}_d(Y_1 \times_k Y_1)$*

$$N\Delta_{Y_1} = \gamma + \gamma_1 + \gamma_2$$

with γ supported in $Z_1 \times Y_1$ for some closed subset $Z_1 \subset Y_1$ of dimension $\leq i$, γ_1 supported on $Y_1 \times D_1$, for some nowhere dense closed subset $D_1 \subset Y_1$, and γ_2 supported in $(Y_1 \cap Y_2) \times Y_1$.

Proof. We consider the (non-proper) \mathcal{O} -scheme $(\mathcal{X} \setminus Y_2) \times_{\mathcal{O}} (\mathcal{X} \setminus Y_2) \rightarrow \text{Spec } \mathcal{O}$, closed subsets Z, D of X with $\dim_K Z \leq i$, D nowhere dense, and a relation

$$N \cdot [\Delta_X] = 0$$

in $\text{CH}_d((X \setminus Z) \times_K (X \setminus D))$, where $[\Delta_X]$ denotes the cycle class represented by the restriction of the diagonal.

As in the proof of Lemma 1.5(1), we have closed subsets Z_0, D_0 of $Y_1^0 := Y_1 \setminus Y_2$ with $\dim_k Z_0 \leq i$, D_0 nowhere dense, and a specialization homomorphism

$$sp : \text{CH}_d((X \setminus Z) \times_K (X \setminus D)) \rightarrow \text{CH}_d((Y_1^0 \setminus Z_0) \times_k (Y_1^0 \setminus D_0)).$$

As X and Y_1^0 are reduced at each generic point, it follows that $sp([\Delta_X]) = [\Delta_{Y_1^0}]$, where $[\Delta_{Y_1^0}]$ is the cycle class of the restriction of the diagonal on Y_1^0 . Applying sp thus gives the relation

$$N \cdot [\Delta_{Y_1^0}] = 0$$

in $\mathrm{CH}_d((Y_1^0 \setminus Z_0) \times (Y_1^0 \setminus D_0))$.

Let $Z_1 := \bar{Z}_0$ be the closures of Z_0 in Y_1 , let \bar{D}_0 be the closure of D_0 in Y_1 and let $D_1 = \bar{D}_0 \cup (Y_1 \cap Y_2)$. Using the localization sequence

$$\begin{aligned} \mathrm{CH}_d(Z_1 \times Y_1 \cup Y_1 \times D_1 \cup (Y_1 \cap Y_2) \times Y_1) \rightarrow \\ \mathrm{CH}_d(Y_1 \times_k Y_1) \rightarrow \mathrm{CH}_d((Y_1^0 \setminus Z_0) \times (Y_1^0 \setminus D_0)) \rightarrow 0 \end{aligned}$$

and the surjection

$$\begin{aligned} \mathrm{CH}_d(Z_1 \times Y_1) \oplus \mathrm{CH}_d(Y_1 \times D_1 \oplus \mathrm{CH}_d((Y_1 \cap Y_2) \times Y_1)) \\ \rightarrow \mathrm{CH}_d(Z_1 \times Y_1 \cup Y_1 \times D_1 \cup (Y_1 \cap Y_2) \times Y_1), \end{aligned}$$

the relation $N \cdot [\Delta_{Y_1^0}] = 0$ in $\mathrm{CH}_d((Y_1^0 \setminus Z_0) \times (Y_1^0 \setminus D_0))$ lifts to a relation of the desired form in $\mathrm{CH}_d(Y_1 \times_k Y_1)$. \square

We conclude this series of specialization results with the following variation on Lemma 1.7; a similar result may be found in [8, Lemma 2.2].

Lemma 1.8. *Let \mathcal{O} be a discrete valuation ring with quotient field K and residue field k . Let $f : \mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}$ be a flat and proper morphism of dimension d over $\mathrm{Spec} \mathcal{O}$ with generic fiber X and special fiber Y . We suppose Y is a union of closed subschemes, $Y = Y_1 \cup Y_2$, with X and Y_1 separable and geometrically irreducible. Suppose that X admits a decomposition of the diagonal of order N . Let $Z = (Y_1 \cap Y_2)_{\mathrm{red}}$ with inclusion $i_Z : Z \rightarrow Y_1$. Suppose further that $Y_{2k(Y_1)}$ admits a zero-cycle y_2 of degree r supported in the smooth locus of $Y_{2k(Y_1)}$.*

Then there is an identity in $\mathrm{CH}_d(Y_1 \times_k Y_1)$

$$Nr\Delta_{Y_1} = \gamma_1 + \gamma_2$$

with γ_1 supported on $Y_1 \times D_1$, for some divisor $D_1 \subset Y_1$, and γ_2 supported in $Z \times Y_1$.

Proof. Let η_1 be the generic point of Y_1 , let $\mathcal{O}_1 = \mathcal{O}_{\mathcal{X}, \eta_1}$ and let \mathcal{D} be the henselization of \mathcal{O}_1 . Let L be the quotient field of \mathcal{D} ; clearly \mathcal{D} has residue field $k(Y_1)$. Then as $\mathrm{Spec} \mathcal{O}_1 \rightarrow \mathrm{Spec} \mathcal{O}$ is essentially smooth, the base-change $\mathcal{X}_{\mathcal{D}} := \mathcal{X} \otimes_{\mathcal{O}} \mathcal{D} \rightarrow \mathrm{Spec} \mathcal{D}$ has generic fiber \mathcal{X}_L and special fiber $Y_{k(Y_1)} = Y_{1k(Y_1)} \cup Y_{2k(Y_1)}$. Let $\mathcal{X}_{\mathcal{D}}^{\mathrm{sm}} \subset \mathcal{X}_{\mathcal{D}}$ be the maximal open subscheme of $\mathcal{X}_{\mathcal{D}}$ that is smooth over \mathcal{D} .

Fix a rational equivalence

$$N \cdot \Delta_X \sim x \times X + \gamma$$

with x a 0-cycle on X and γ supported on $X \times E$ for some divisor E . Pulling this back to X_L gives the rational equivalence

$$N \cdot \Delta_{X_L} \sim x_L \times_L X_L + \gamma_L$$

with γ_L supported on $X_L \times_L E_L$. Let \mathcal{E} be the closure of E_L in $\mathcal{X}_{\mathcal{D}}$ and let $E_0 = \mathcal{E} \cap Y_{k(Y_1)}$; E_0 contains no generic point of $Y_{k(Y_1)}$. Furthermore, since the 0-cycle y_2 on $Y_{2k(Y_1)}$ is contained in the smooth locus of $Y_{2k(Y_1)}$, we may find a 0-cycle y'_2 on $Y_{2k(Y_1)}$, rationally equivalent to y_2 , and with support in the smooth locus of $Y_{2k(Y_1)} \setminus (E_0 \cup Z_{k(Y_1)})$. Changing notation, we may assume that y_2 is supported in the smooth locus of $Y_{2k(Y_1)} \setminus (E_0 \cup Z_{k(Y_1)})$.

Since \mathcal{D} is Hensel, we may lift $\eta_1 \in Y_1(k(Y_1))$ to a section $s_1 : \text{Spec } \mathcal{D} \rightarrow \mathcal{X}_{\mathcal{D}}$. Since y_2 is supported in the smooth locus of $Y_{k(Y_1)}$, we may similarly lift the 0-cycle y_2 on $Y_{2k(Y_1)}$ to a cycle η_2 on $\mathcal{X}_{\mathcal{D}}$ of relative dimension zero and relative degree r over \mathcal{D} . This gives us the 0-cycle of degree zero $\rho_L := r \cdot s_1(\text{Spec } L) - \eta_{2L}$ on X_L . Since \mathcal{D} is local, $\mathcal{X}_{\mathcal{D}}$ is flat over \mathcal{D} and both y_2 and η_1 are supported in the smooth locus of $Y \setminus E_0$, it follows that both $s_1(\text{Spec } \mathcal{D})$ and η_2 are supported in $\mathcal{X}_{\mathcal{D}}^{sm} \setminus \mathcal{E}$, and thus ρ_L is supported in the smooth locus of $X_L \setminus E$.

Let p be a closed point in the smooth locus of X_L , inducing the inclusion $i_p : X_L \times_L p \rightarrow X_L \times_L X_L$. Since i_p is a regular codimension $d = \dim X$ embedding, we have the pull-back map (see [11, Chap. 6])

$$i_p^* : \text{CH}_d(X_L \times_L X_L) \rightarrow \text{CH}_0(X_L \times_L p)$$

If \mathfrak{z} is a 0-cycle supported in the smooth locus of X_L , $\mathfrak{z} = \sum_j n_j p_j$, we have the map

$$\mathfrak{z}^* : \text{CH}_d(X_L \times_L X_L) \rightarrow \text{CH}_0(X_L)$$

defined as the sum $\sum_j n_j p_{1*} \circ i_{p_j}^*$. If γ is a d -cycle on $X_L \times_L X_L$ such that each component of γ intersects each subvariety $X_L \times p_j$ properly, then $\gamma^*(\mathfrak{z})$ is well-defined and

$$\mathfrak{z}^*(\gamma) = \gamma^*(\mathfrak{z}).$$

We apply these comments to the 0-cycle ρ_L and the cycles $N \cdot \Delta_{X_L}$, $x_L \times_L X_L$ and γ_L . We get the identities in $\text{CH}_0(X_L)$

$$\begin{aligned} N \cdot \rho_L &= \rho_L^*(N \cdot \Delta_{X_L}) \\ &= \rho_L^*(x_L \times_L X_L) + \rho_L^*(\gamma_L). \end{aligned}$$

Both terms in this last line are zero, the first since, as X_L is irreducible, we have $\rho_L^*(x_L \times_L X_L) = \deg(\rho_L) \cdot x_L = 0$, and the second since $X_L \times \text{supp}(\rho_L) \cap \text{supp}(\gamma_L) = \emptyset$. In other words, $N \cdot \rho_L = 0$ in $\text{CH}_0(X_L)$.

We apply the specialization map

$$sp : \text{CH}_0(X_L) \rightarrow \text{CH}_0(Y_{k(Y_1)})$$

and find that $N(r \cdot \eta_1 - y_2) = 0$ in $\text{CH}_0(Y_{k(Y_1)})$. Thus $Nr \cdot \eta_1 = 0$ in $\text{CH}_0(Y_{1k(Y_1)} \setminus Z_{k(Y_1)})$, and by using the localization sequence for the inclusion

$Z_{k(Y_1)} \rightarrow Y_{k(Y_1)}$, there is a 0-cycle $\gamma_{2k(Y_1)}$ on $Z_{k(Y_1)}$ with

$$Nr \cdot \eta_1 = i_{Z*}(\gamma_{2k(Y_1)})$$

in $\text{CH}_0(Y_{1k(Y_1)})$. Spreading this relation out over Y_1 as in previous proofs gives the desired decomposition of $Nr \cdot \Delta_{Y_1}$. \square

Remark 1.9. Suppose we have \mathcal{X} , $Y = Y_1 \cup Y_2$ and $Z = Y_1 \cap Y_2$ satisfying the hypotheses of Lemma 1.8; suppose in addition that Y_1 is smooth over k . Then for all fields $F \supset k$, the quotient group $\text{CH}_0(Y_{1F})/i_{Z*}(\text{CH}_0(Z_F))$ is Nr -torsion. Indeed, since Y_1 is smooth, we have an operation of correspondences on $\text{CH}_0(Y_{1F})$, the correspondence γ_1^* of Lemma 1.8 acts trivially on $\text{CH}_0(Y_{1F})$, γ_2^* maps $\text{CH}_0(Y_{1F})$ to $i_{Z*}(\text{CH}_0(Z_F))$ and the sum acts by multiplication by Nr .

The torsion orders behave well with respect to base-change.

Lemma 1.10. *Let X and Y be proper separable k -schemes, with Y integral and with X equi-dimensional over k . Let K be the function field $k(Y)$, I_Y the index of Y .*

1. *For all i , $\text{Tor}_k^{(i)}(X)$ is finite if and only if $\text{Tor}_K^{(i)}(X_K)$ is finite and in this case, $\text{Tor}_k^{(i)}(X)$ divides $I_Y \text{Tor}_K^{(i)}(X_K)$.*
2. *Suppose X is geometrically integral. If $g\text{Tor}_k(X)$ is finite, then so is $\text{Tor}_k(X)$ and $\text{Tor}_k(X)$ divides $I_X \cdot g\text{Tor}_k(X)$.*
3. *Let $k \subset L$ be an extension of fields with k algebraically closed. Then $\text{Tor}_k^{(i)}(X) = \text{Tor}_L^{(i)}(X_L)$ for all i . Suppose in addition X is smooth and integral. Then $g\text{Tor}_k(X) = g\text{Tor}_L(X_L)$ and $\text{Tor}_k(X) = g\text{Tor}_k(X)$.*

Proof. (1) If $\text{Tor}_k^{(i)}(X)$ is finite, then so is $\text{Tor}_K^{(i)}(X_K)$ by Lemma 1.3(4). Suppose $\text{Tor}_K^{(i)}(X_K)$ is finite. Let y be a closed point of Y , contained in the smooth locus of Y over k , and let $\mathcal{O} := \mathcal{O}_{Y,y}$. Applying Lemma 1.5 to the constant family $\mathcal{X} := X \times_k \mathcal{O}$, we see that $\text{Tor}_{k(y)}^{(i)}(X_{k(y)})$ is finite and $\text{Tor}_{k(y)}^{(i)}(X_{k(y)})$ divides $\text{Tor}_K^{(i)}(X_K)$. Applying Lemma 1.3(4) again, $\text{Tor}_k^{(i)}(X)$ is finite and divides $[k(y) : k] \cdot \text{Tor}_{k(y)}^{(i)}(X_{k(y)})$. This proves the first assertion.

For (2), let y be a closed point of X , contained in the smooth locus of X over k , let $\mathcal{O} := \mathcal{O}_{X,y}$, and let $\eta \in X(k(X))$ be the canonical point, that is, the restriction of the diagonal section $X \rightarrow X \times_k X$ to $\text{Spec } k(X)$. As in the proof of Lemma 1.5, we may stratify $\text{Spec } \mathcal{O}$ by regular closed subschemes $y = Z_0 \subset \dots \subset Z_d = \text{Spec } \mathcal{O}$, $d = \dim_k X$, and thereby define specialization homomorphisms

$$sp_i : \text{CH}_0(X_{k(Z_i)(X)}) \rightarrow \text{CH}_0(X_{k(Z_{i-1})(X)}); \quad i = 1, \dots, d.$$

Letting $sp_y : \text{CH}_0(X_{k(X \times_k X)}) \rightarrow \text{CH}_0(X_{k(y)(X)})$ be the composition of the sp_i , we have $sp_y(\eta_1 - \eta_2) = \eta_y - y_{gen}$, where $\eta_y \in X(k(y)(X))$ is base-change of $y \in X(k(y))$ and $y_{gen} \in X(k(y)(X))$ is the base-change of $\eta \in X(k(X))$. Thus $g\text{Tor}_k(X) \cdot (\eta_y - y_{gen}) = 0$ in $\text{CH}_0(X_{k(y)(X)})$; pushing forward

to $\mathrm{CH}_0(X_{k(X)})$ gives $[k(y) : k] \cdot g\mathrm{Tor}_k(X) \cdot \eta - g\mathrm{Tor}_k(X) \cdot y \times_k k(X) = 0$ in $\mathrm{CH}_0(X_{k(X)})$. Applying localization gives us the decomposition of the diagonal Δ_X of order $[k(y) : k] \cdot g\mathrm{Tor}_k(X)$; doing this for each closed point y gives us the decomposition of the diagonal of order $I_X \cdot g\mathrm{Tor}_k(X)$, hence $\mathrm{Tor}_k(X)$ is finite and divides $I_X \cdot g\mathrm{Tor}_k(X)$.

For (3), we may assume that L is finitely generated over k , so that $L = k(Y)$ for some integral proper k -scheme Y . Since k is algebraically closed, $I_Y = 1$, so the first assertion for $\mathrm{Tor}^{(i)}$ follows from (1). The assertions about $g\mathrm{Tor}$ follow from this, (2) and Lemma 1.3. \square

For example, $\mathrm{Tor}_k^{(i)}(X) = \mathrm{Tor}_L^{(i)}(X_L)$ if L is a pure transcendental extension of a field k .

Definition 1.11. Let X be a proper, separable k -scheme. Let \bar{k} be the algebraic closure of k and define $\mathrm{Tor}^{(i)}(X) := \mathrm{Tor}_{\bar{k}}^{(i)}(X_{\bar{k}})$. We call $\mathrm{Tor}^{(i)}(X)$ the i th *geometric torsion order* of X . We write $\mathrm{Tor}(X)$ for $\mathrm{Tor}^{(0)}(X)$.

Note that $\mathrm{Tor}^{(i)}(X)$ is invariant under base-extension $X \rightsquigarrow X_L$ for a field extension $L \supset k$. Also, assuming X to be smooth and geometrically integral, $\mathrm{Tor}(X)$ is equal to $g\mathrm{Tor}_{\bar{k}}(X_{\bar{k}})$.

In much the same vein as Lemma 1.3, we show that the generic torsion order measures the torsion order after adjoining a “generic” rational point, that is:

Lemma 1.12. *Let X be a smooth proper geometrically integral k -scheme and let $K = k(X)$. Then $g\mathrm{Tor}_k(X) = \mathrm{Tor}_K(X_K)$.*

Proof. If $N \cdot (\eta_1 - \eta_2) = 0$ in $\mathrm{CH}_0(X_{k(X \times_k X)})$, then we have a decomposition of the diagonal of order N for $X_{k(X)}$:

$$N \cdot \Delta_{X_K} = N \cdot \eta \times_K X_K + \gamma$$

with γ supported in $X_K \times_K D$, with $D \subsetneq X_K$, and with η the restriction of Δ_X to $X \times_k k(X) \subset X \times_k X$. Thus $\mathrm{Tor}_K(X_K)$ divides $g\mathrm{Tor}_k X$. Conversely, if X_K admits a decomposition of the diagonal of order n ,

$$n \cdot \Delta_{X_K} = x \times X_K + \gamma$$

with x a 0-cycle on X_K and γ supported on $X_K \times D$ for some divisor $D \subset X_K$, then applying $n \cdot \Delta_{X_K}^*$ to η gives us $x = n \cdot \eta$ in $\mathrm{CH}_0(X_K)$, so $n \cdot \Delta_{X_K} = n \cdot \eta \times X_K + \gamma$ in $\mathrm{CH}_d(X_K \times_K X_K)$. Restriction to $X \times_K K(X_K)$ gives $n \cdot \eta_1 = n \cdot \eta_2$ in $\mathrm{CH}_0(X_{k(X \times_k X)})$, so $g\mathrm{Tor}_k(X)$ divides $\mathrm{Tor}_K(X_K)$. \square

One last elementary property of the torsion indices concerns the behavior with respect to morphisms

Lemma 1.13. *Let $f : Y \rightarrow X$ be a surjective morphism of integral reduced proper k -schemes of the same dimension d . Then $\mathrm{Tor}_k^{(i)} X$ divides $\deg f \cdot \mathrm{Tor}_k^{(i)} Y$ for all i . If X and Y are separable over k , then $g\mathrm{Tor}_k X$ divides $(\deg f)^2 \cdot g\mathrm{Tor}_k Y$.*

Proof. Suppose the diagonal for Y admits a decomposition of order N and level i :

$$N \cdot \Delta_Y = \gamma_i + \gamma'$$

with γ' supported on $Y \times D$ for some divisor D and γ_i supported on $Z \times Y$ for some closed subset Z of Y with $\dim_k Z \leq i$. Pushing forward by $f \times f$ gives

$$\deg f \cdot N \cdot \Delta_X = (f \times f)_* \gamma_i + (f \times f)_* \gamma',$$

and thus $\text{Tor}_k^{(i)} X$ divides $\deg f \cdot \text{Tor}_k^{(i)} Y$. Similarly, we have $(f \times f \times f)_*(\Delta_{Y,ij}) = (\deg f)^2 \cdot \Delta_{X,ij}$ for $ij = 12, 13$, which shows that $g\text{Tor}_k X$ divides $(\deg f)^2 \cdot g\text{Tor}_k Y$. \square

The behavior of the torsion indices with respect to rational and birational maps will be discussed in the next section.

2. UNIVERSALLY AND TOTALLY CH_0 -TRIVIAL MORPHISMS

We recall the notion of a universally CH_0 -trivial morphism and a related notion, that of a totally CH_0 -trivial morphism.

Definition 2.1 ([6, Definitions 1.1, 1.2]). Let $p : Z \rightarrow Y$ be a proper morphism of finite type k -schemes for some field k . The morphism p is *universally CH_0 -trivial* if for all field extensions $F \supset k$, the map $p_* : \text{CH}_0(Z_F) \rightarrow \text{CH}_0(Y_F)$ is an isomorphism. A proper k -scheme $\pi_Y : Y \rightarrow \text{Spec } k$ is called a *universally CH_0 -trivial k -scheme* if π_Y is a universally CH_0 -trivial morphism.

Definition 2.2. A proper morphism $p : Z \rightarrow Y$ of k -schemes is *totally CH_0 -trivial* if for each point $y \in Y$, the fiber $p^{-1}(y)$ is a universally CH_0 -trivial $k(y)$ -scheme.

It follows directly from the definition that the property of a proper morphism being totally CH_0 -trivial is stable under arbitrary base-change.

We rephrase a result of Colliot-Thélène and Pirutka.

Proposition 2.3 ([6, Proposition 1.7]). *Let $p : Z \rightarrow Y$ be a totally CH_0 -trivial morphism. Then p is universally CH_0 -trivial.*

Remarks 2.4. 1. By the base-change property of totally CH_0 -trivial morphisms, we see that for $p : Z \rightarrow Y$ a totally CH_0 -trivial morphism and $W \rightarrow Y$ a morphism of k -schemes, the projection $Z \times_Y W \rightarrow W$ is universally CH_0 -trivial.

2. There are examples of universally CH_0 -trivial morphisms that are not

totally CH_0 -trivial¹; in particular, the property of a morphism being universally CH_0 -trivial is not stable under base-change.

Corollary 2.5. *1. Universally CH_0 -trivial morphisms and totally CH_0 -trivial morphisms are closed under composition.*

2. Let $p : Z \rightarrow Y$ be a morphism of smooth k -schemes that is a sequence of blow-ups with smooth centers. Then p is a totally CH_0 -trivial morphism.

3. Suppose that the field k admits resolution of singularities of birational morphisms for smooth k -schemes of dimension $\leq d$, that is: if $p : Z \rightarrow Y$ is a proper birational morphism of smooth k -schemes of dimension $\leq d$, there is a sequence of blow-ups of Y with smooth centers, $q : W \rightarrow Y$, such that resulting birational map $r : W \rightarrow Z$ is a morphism. Then each proper birational morphism $p : Z \rightarrow Y$ of smooth k -schemes of dimension $\leq d$ is totally CH_0 -trivial. In particular, this holds for k of characteristic zero, or for $d \leq 3$ and k algebraically closed (see [1]).

Proof. (1) for universally CH_0 -trivial morphisms is obvious from the definition and for totally CH_0 -trivial morphisms this follows with the help of Proposition 2.3.

For (2), we use (1) to reduce to checking for the blow-up of Y along a smooth closed subscheme F , for which the assertion is clear.

For (3), let y be a point of Y and $L \supset k(y)$ a field extension. Dominating Z by a $q : W \rightarrow Y$ as above, we have the maps

$$\text{CH}_0(q^{-1}(y)_L) \xrightarrow{r_*} \text{CH}_0(p^{-1}(y)_L) \xrightarrow{p_*} \text{CH}_0(\text{Spec } L) = \mathbb{Z}$$

which, as $\text{CH}_0(q^{-1}(y)_L) \rightarrow \text{CH}_0(\text{Spec } L)$ is an isomorphism, gives us a splitting to p_* . Applying resolution of singularities to $r : W \rightarrow Z$ gives a sequence of blow-ups with smooth centers $s : X \rightarrow Z$ such that $t := r^{-1}s : X \rightarrow W$ is a morphism. Since $X \rightarrow Z$ is totally CH_0 -trivial, the sequence

$$\text{CH}_0(t^{-1}(q^{-1}(y))_L) \xrightarrow{t_*} \text{CH}_0(q^{-1}(y)_L) \xrightarrow{r_*} \text{CH}_0(p^{-1}(y)_L)$$

gives a splitting to r_* , so p_* is an isomorphism. \square

Lemma 2.6. *1. Let $q : Z \rightarrow Y$ be a birational totally CH_0 -trivial morphism of integral, separable, k -schemes. Let $N > 0$ be an integer, let $Y_i, W, D \subset Y$ be proper closed subsets with $\dim Y_i \leq i$, and suppose we have a decomposition of Δ_Y as*

$$N \cdot \Delta_Y = \gamma + \gamma_1 + \gamma_2,$$

¹For example, let k be an algebraically closed field of characteristic $\neq 2$, let S be the cone in \mathbb{P}_k^3 over a smooth plane curve C of degree ≥ 3 , let $Y \rightarrow S$ be the double cover branched over the transverse intersection of S with a quadric, and let $y_1, y_2 \in Y$ be the points lying over the vertex of S . Let $p : Z \rightarrow Y$ be the blow-up of Y at y_1 and let $z = p^{-1}(y_2)$. Then for all fields $L \supset k$, $\text{CH}_0(z_L) \xrightarrow{iz_*} \text{CH}_0(Z_L)$ and $\text{CH}_0(y_{2L}) \xrightarrow{iy_{2*}} \text{CH}_0(Y_L)$ are isomorphisms, and thus p is universally CH_0 -trivial. However, $p^{-1}(y_1) \cong C$, so p is not totally CH_0 -trivial.

with γ supported on $Y_i \times Y$, γ_1 supported on $Y \times D$ and γ_2 supported on $W \times Y$. Then there are proper closed subsets $Z_i, D' \subset Z$ with $\dim Z_i \leq i$ and a decomposition of Δ_Z as

$$N \cdot \Delta_Z = \gamma' + \gamma'_1 + \gamma'_2,$$

with γ' supported on $Z_i \times Z$, γ'_1 supported on $Z \times D'$ and γ'_2 supported on $q^{-1}(W) \times Z$.

2. Let $q : Z \rightarrow Y$ be a birational totally CH_0 -trivial morphism of integral, separable, proper k -schemes. Then $\text{Tor}_k^{(i)}(Z) = \text{Tor}_k^{(i)}(Y)$ for all i .
3. Let $q : Z \rightarrow Y$ be a birational universally CH_0 -trivial morphism of integral proper k -schemes. Then $\text{Tor}_k(Z) = \text{Tor}_k(Y)$. If moreover Z and Y are geometrically integral, then $g\text{Tor}_k(Z) = g\text{Tor}_k(Y)$.

Proof. We note that (2) follows easily from (1). Indeed, (1) with $W = \emptyset$ shows that $\text{Tor}_k^{(i)}(Z)$ divides $\text{Tor}_k^{(i)}(Y)$ for all i ; as $(q \times q)_*(\Delta_Z) = \Delta_Y$, it follows that a decomposition of Δ_Z of order N and level i gives a similar decomposition of Δ_Y by applying $(q \times q)_*$.

We now prove (1). We may assume that $W = \emptyset$. Indeed, if we replace Y with $Y' := Y \setminus W$ and Z with $Z' := Z \setminus q^{-1}(W)$, the result for $q|_{Z'} : Z' \rightarrow Y'$ and the decomposition

$$N \cdot \Delta_{Y'} = \gamma|_{Y' \times Y'} + \gamma_1|_{Y' \times Y'},$$

together with localization gives (1) for the original data.

Suppose then we have

$$N \cdot \Delta_Y = \gamma + \gamma_1$$

with γ supported on $Y_i \times Y$ and γ_1 supported on $Y \times D$. Let $K = k(Y)$ and let $\eta_Y \in Y$ be the generic point. We have a rational equivalence of 0-cycles on $Y \times \eta_Y$

$$N \cdot \eta_Y \times \eta_Y \sim \gamma_{\eta_Y}$$

with γ_{η_Y} a 0-cycle supported on $Y_i \times \eta_Y$. Thus $N \cdot \eta_Y \times \eta_Y \sim 0$ on $(Y \setminus Y_i) \times \eta_Y$.

Since $Z \setminus q^{-1}(Y_i) \rightarrow Y \setminus Y_i$ is birational and universally CH_0 -trivial (Remark 2.4), there is a rational equivalence of 0-cycles

$$N \cdot \eta_Z \times \eta_Z \sim 0$$

on $(Z \setminus q^{-1}(Y_i)) \times \eta_Z$, where $\eta_Z \in Z$ is the generic point. We claim that there is a dimension $\leq i$ closed subset Z' of Z and a rational equivalence of 0-cycles on $Z \times \eta_Z$

$$N \cdot \eta_Z \times \eta_Z \sim \rho_Z$$

with ρ_Z a 0-cycle supported on $Z' \times \eta_Z$. We proceed by a noetherian induction: We assume there is a closed subset $Y^j \subset Y_i$, a dimension $\leq i$ closed subset Z_j of $q^{-1}(Y_i)$ and a rational equivalence of 0-cycles on $(Z \setminus q^{-1}(Y^j)) \times \eta_Z$

$$N \cdot \eta_Z \times \eta_Z \sim \rho_j$$

with ρ_j a 0-cycle supported on $Z_j \times \eta_Z$, and we show the parallel statement for a proper closed subset Y^{j+1} of Y^j . The induction starts with $Y^0 = Y_i$.

Chose an irreducible component Y_0^j of Y^j and let ν be its generic point. Let Y' be the union of the components of Y^j different from Y_0^j . We have the exact localization sequence

$$\begin{aligned} \mathrm{CH}_0((q^{-1}(Y_0^j \setminus Y')) \times \eta_Z) &\xrightarrow{i_*} \mathrm{CH}_0((Z \setminus q^{-1}(Y')) \times \eta_Z) \\ &\rightarrow \mathrm{CH}_0((Z \setminus q^{-1}(Y^j)) \times \eta_Z) \rightarrow 0 \end{aligned}$$

and thus there is a 0-cycle ρ' on $q^{-1}(Y_0^j \setminus Y') \times \eta_Z$ and a rational equivalence

$$N \cdot \eta_Z \times \eta_Z \sim \rho_j + i_*(\rho')$$

on $(Z \setminus q^{-1}(Y')) \times \eta_Z$.

Write

$$\rho' = \sum_i m_i x_i + \sum_j n_j x'_j,$$

where the x_i, x'_j are closed points of $q^{-1}(Y_0^j \setminus Y') \times \eta_Z$, such that $q \circ p_1(x_i) = \nu$ for all i and $q \circ p_1(x'_j)$ is contained in some proper closed subset (say Y'') of Y_0^j for all j . Replacing Y' with $Y' \cup Y''$ and changing notation, we may assume that $\rho' = \sum_i m_i x_i$.

By assumption, the map $q^{-1}(\nu) \rightarrow \nu$ is universally CH_0 -trivial, so there is a degree one 0-cycle ϵ on $q^{-1}(\nu)$ so that ϵ_L generates $\mathrm{CH}_0(q^{-1}(\nu)_L)$ for all field extensions $L \supset k(\nu)$, in particular, $\epsilon \times \eta_Z$ generates $\mathrm{CH}_0(q^{-1}(\nu) \times \eta_Z)$. Enlarging Y' again by a proper closed subset of Y_0^j , we may assume that

$$\rho' = m \cdot \epsilon \times \eta_Z$$

in $\mathrm{CH}_0(q^{-1}(Y_0^j \setminus Y') \times \eta_Z)$, for some $m \in \mathbb{Z}$. Since ϵ is a 0-cycle on $q^{-1}(\nu)$, the closure Z' of the support of ϵ in $q^{-1}(Y_0^j)$ has dimension over k bounded by the transcendence dimension of $k(\nu)$ over k , that is, by $\dim_k Y_0^j$; since $Y_0^j \subset Y_i$, we have

$$\dim_k Z' \leq i.$$

Taking $Y^{j+1} = Y'$, $Z_{j+1} = Z_j \cup Z'$, $\rho_{j+1} = \rho_j + m \cdot \epsilon \times \eta_Z$, the 0-cycle ρ_{j+1} is supported on $Z_{j+1} \times \eta_Z$, $\dim_k Z_{j+1} \leq i$, and we have

$$N \cdot \eta_Z \times \eta_Z = \rho_{j+1}$$

in $\mathrm{CH}_0((Z \setminus q^{-1}(Y^{j+1})) \times \eta_Z)$. The induction thus goes through, proving the result.

The proof of (3) is similar but easier. We have already seen that if Z has a decomposition of the diagonal of order N , then so does Y . If conversely Y has a decomposition of the diagonal of order N , then there is a 0-cycle y on Y with

$$N \cdot \eta_Y \times \eta_Y = y \times \eta_Y$$

in $\mathrm{CH}_0(Y \times \eta_Y)$. As $q : Z \rightarrow Y$ is universally CH_0 trivial, there is a 0-cycle z on Z with $q_* z = y$ in $\mathrm{CH}_0(Y)$ and since $(q \times q)_* : \mathrm{CH}_0(Z \times \eta_Z) \rightarrow \mathrm{CH}_0(Y \times \eta_Y)$ is an isomorphism, we have

$$N \cdot \eta_Z \times \eta_Z = z \times \eta_Z$$

in $\mathrm{CH}_0(Z \times \eta_Z)$. The proof for $g\mathrm{Tor}$ is the same. \square

We note some consequences of Lemma 2.6.

Proposition 2.7. *Let $f : Y \rightarrow X$ be a dominant rational map of smooth integral proper k -schemes of the same dimension d .*

1. *Suppose k admits resolution of singularities for rational maps of varieties of dimension $\leq d$, that is, if $p : Y \rightarrow X$ is a rational morphism of smooth k -schemes of dimension $\leq d$, there is a sequence of blow-ups of Y with smooth center, $q : W \rightarrow Y$, such that resulting rational map $r : W \rightarrow X$ is a morphism. Then $\mathrm{Tor}_k^{(i)} X$ divides $\deg f \cdot \mathrm{Tor}_k^{(i)} Y$ for all i .*
2. *Without assumption on k , $\mathrm{Tor}_k X$ divides $\deg f \cdot \mathrm{Tor}_k Y$ and $g\mathrm{Tor}_k X$ divides $(\deg f)^2 \cdot g\mathrm{Tor}_k Y$.*

Proof. For (1) we may find a sequence of blow-ups with smooth centers, $g : Z \rightarrow Y$, so that the induced rational map $h : Z \rightarrow X$ is a morphism. Since g is a totally CH_0 -trivial morphism, $\mathrm{Tor}_k^{(i)} Z = \mathrm{Tor}_k^{(i)} Y$ by Lemma 2.6(2), so we may assume that g is a morphism; the result then follows from Lemma 1.13.

For (2), let $Z \subset Y \times X$ be the graph of f , that is, the closure of the graph of $f : V \rightarrow X$ for a non-empty open subset $V \subset Y$ on which f is defined. The map $p_1 : Z \rightarrow Y$ is birational and there is a non-empty open $X_0 \subset X$ such that $p_1 : p_2^{-1}(X_0) \cap Z \rightarrow Y$ is an open immersion; set $Y_0 := p_1(p_2^{-1}(X_0) \cap Z)$. The correspondence $Z \times_k Z$ yields a homomorphism

$$g : \mathrm{CH}_d(Y \times Y) \rightarrow \mathrm{CH}_d(X \times X).$$

We claim that $g(\Delta_Y) = \deg(f) \cdot \Delta_X + \gamma$ where γ is a cycle supported on $X \times (X \setminus X_0)$, which implies the assertion for $\mathrm{Tor}_k X$. Keeping track of supports and using localization, we have an identity in $\mathrm{CH}_d(Z \times_k Z)$ of the form

$$(2.1) \quad [Z \times Z] \cdot (p_1 \times p_1)^*(\Delta_Y) = \Delta_Z + \gamma',$$

where γ' has support in $(p_1^{-1}(Y \setminus Y_0) \cap Z) \times_{Y \setminus Y_0} (p_1^{-1}(Y \setminus Y_0) \cap Z)$. Thus $(p_2 \times p_2)_*(\gamma')$ has support in $X \times (X \setminus X_0)$. Applying $(p_2 \times p_2)_*$ to (2.1) we prove our claim.

The proof for $g\mathrm{Tor}_k$ is similar. \square

In particular, if we have resolution of singularities of birational maps, $\mathrm{Tor}_k^{(i)}$ is a birational invariant and in general Tor_k is a birational invariant; from this it follows easily that $\mathrm{Tor}_k^{(i)}$ is a stable birational invariant if we have resolution of singularities of birational maps and in general Tor_k is a stable birational invariant.

3. SPECIALIZATION AND DEGENERATION

The next result, in a somewhat different form, is proven in [6, Théorème 1.12]. In a less general setting, a similar result may be found in [29, Theorem 1.1].

Proposition 3.1. *Let \mathcal{O} be a regular local ring with quotient field K and residue field k . Let $f : \mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}$ be a flat and proper morphism with geometrically integral fibers and let X be the generic fiber \mathcal{X}_K , Y the special fiber \mathcal{X}_k . We suppose that Y admit a resolution of singularities $q : Z \rightarrow Y$ such that q is a universally CH_0 -trivial morphism. Suppose in addition that X admits a decomposition of the diagonal of order N . Then Z also admits a decomposition of the diagonal of order N . In particular, if $\operatorname{Tor}_K(X)$ is finite then so is $\operatorname{Tor}_k(Z)$, and in this case $\operatorname{Tor}_k(Z) \mid \operatorname{Tor}_K(X)$.*

In [6] it is assumed that X has a resolution of singularities $\tilde{X} \rightarrow X$ such that \tilde{X}_K admits a decomposition of the diagonal of order N , which implies the same condition on X by pushing forward; there is also an assumption that Z has a 0-cycle of degree 1. This resolution of singularities in [6] arises because they consider decompositions of the diagonal only on smooth proper varieties; the existence of a degree 1 0-cycle comes from considering only the case $N = 1$. The modified version stated above is proved exactly as in *loc cit.*

We prove an extension of this specialization result which takes the decompositions of higher level into account.

Proposition 3.2. *Let \mathcal{O} be a regular local ring with quotient field K and residue field k . Let $f : \mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}$ be a flat and proper morphism with geometrically integral fibers and let X be the generic fiber \mathcal{X}_K , Y the special fiber \mathcal{X}_k . Suppose that there is a birational totally CH_0 -trivial morphism $q : Z \rightarrow Y$ of geometrically integral proper k -schemes.*

1. *Suppose X admits a decomposition of the diagonal of order N and level i . Then Z also admits a decomposition of the diagonal of order N and level i . If $\operatorname{Tor}_K^{(i)}(X)$ is finite then so is $\operatorname{Tor}_k^{(i)}(Z)$ and in this case $\operatorname{Tor}_k^{(i)}(Z) \mid \operatorname{Tor}_K^{(i)}(X)$.*
2. *Let \bar{K} and \bar{k} be the respective algebraic closures of K and k and suppose that $X_{\bar{K}}$ admits a decomposition of the diagonal of order N and level i . Suppose that K has characteristic zero, or that \mathcal{O} is excellent. Then $Z_{\bar{k}}$ also admits a decomposition of the diagonal of order N and level i . If $\operatorname{Tor}^{(i)}(X)$ is finite then so is $\operatorname{Tor}^{(i)}(Z)$ and in this case $\operatorname{Tor}^{(i)}(Z) \mid \operatorname{Tor}^{(i)}(X)$.*

Proof. The assertion (2) follows from (1) by first stratifying $\operatorname{Spec} \mathcal{O}$ as in the proof of Lemma 1.5 to reduce to the case of a DVR. We then take a finite extension L of K so that $\operatorname{Tor}^{(i)}(X) = \operatorname{Tor}_L^{(i)}(X_L)$, take the normalization $\mathcal{O} \rightarrow \mathcal{O}^N$ of \mathcal{O} in L and replace \mathcal{O} with the localization \mathcal{O}' of \mathcal{O}^N at some maximal ideal. Letting k' be the residue field of \mathcal{O}' , $\operatorname{Tor}^{(i)}(Z)$ divides $\operatorname{Tor}_{k'}^{(i)}(Z_{k'})$, so (1) implies (2). We now prove (1).

By Lemma 1.5, Y admits a decomposition of the diagonal of order N and level i . By Lemma 2.6, Z also admits a decomposition of the diagonal of order N and level i , proving (1). \square

We also have a version that incorporates Totaro's extended specialization Lemma 1.7.

Proposition 3.3. *Let \mathcal{O} be a discrete valuation ring with quotient field K and residue field k . Let $f : \mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}$ be a flat and proper morphism of dimension d over $\operatorname{Spec} \mathcal{O}$ with generic fiber X and special fiber Y . We suppose Y is a union of closed subschemes, $Y = Y_1 \cup Y_2$ and that X and Y_1 are geometrically integral. Suppose there is a birational totally CH_0 -trivial morphism $q : Z \rightarrow Y_1$ of geometrically integral proper k -schemes and that X admits a decomposition of the diagonal of order N and level i . Then there are proper closed subsets $Z_i, D \subset Z$ with $\dim Z_i \leq i$ and a decomposition*

$$N \cdot \Delta_Z = \gamma + \gamma_1 + \gamma_2$$

with γ supported in $Z_i \times Z$, γ_1 supported in $Z \times D$ and γ_2 supported in $q^{-1}(Y_1 \cap Y_2) \times Z$.

Proof. This follows directly from Lemma 1.7 and Lemma 2.6. \square

Remark 3.4. As in the second part of Proposition 3.2, we may take the N in Proposition 3.3 to be $\operatorname{Tor}_K^{(i)}(X_{\bar{K}})$ if \mathcal{O} is excellent or if K has characteristic zero, by replacing \mathcal{O} with its normalization \mathcal{O}' in a finite extension L of K so that $\operatorname{Tor}_K^{(i)}(X_{\bar{K}}) = \operatorname{Tor}_L^{(i)}(X_L)$, replacing \mathcal{X} with $\mathcal{X} \times_{\mathcal{O}} \mathcal{O}'$, replacing k with the residue field k' of \mathcal{O}' and replacing Z with $Z \otimes_k k'$.

4. TORSION ORDER FOR COMPLETE INTERSECTIONS IN A PROJECTIVE SPACE: AN UPPER BOUND

We concentrate on the 0th torsion order of a (reduced, separable) complete intersection $X = X_{d_1, \dots, d_r}^n$ in \mathbb{P}^{n+r} of dimension n and multi-degree d_1, d_2, \dots, d_r . In this section, we recall the construction of Roitman [25], which gives an upper bound for $\operatorname{Tor}_k(X)$; by Lemma 1.3(1), this gives an upper bound for $\operatorname{Tor}_k^{(i)}(X)$ for all i .

We often shorten the notation by writing d_* for a sequence d_1, d_2, \dots, d_r .

Proposition 4.1. *Let k be a field and let $X = X_{d_1, \dots, d_r}^n$ in \mathbb{P}_k^{n+r} with $\sum_i d_i \leq n+r$ be a reduced, separable complete intersection of multi-degree d_1, \dots, d_r , with $n \geq 1$. Then $\operatorname{Tor}_k(X)$ is finite and divides $\prod_{i=1}^r d_i!$.*

Proof. The reduced, separable complete intersections in \mathbb{P}^{n+r} and of multi-degree d_1, \dots, d_r are parametrized by an open subscheme $\mathcal{U}_{d_*;n}$ of a product of projective spaces; by Lemma 1.5 it suffices to prove the result for the subscheme $X := X_{d_*, \text{gen}}$ of \mathbb{P}_K^{n+r} defined over the field $K := k(\mathcal{U}_{d_*;n})$ corresponding to the generic point of $\mathcal{U}_{d_*;n}$. For such an X , there is an open subset $V \subset X$, such that, for $x \in V$, the set of lines $\ell \subset \mathbb{P}^{n+r}$ such that

$x \in \ell$ and $(\ell \cap X)_{\text{red}}$ is either $\{x\}$ or ℓ is defined by a complete intersection W_x of multi-degree

$$d_1 - 1, d_1 - 2, \dots, 2, 1, d_2 - 1, d_2 - 2, \dots, 2, 1, \dots, d_r - 1, \dots, 2, 1$$

in the projective space $\mathbb{P}_{K(x)}^{n+r-1}$ of lines through x . Indeed, we may choose a standard affine open U in $\mathbb{P}_{K(x)}^{n+r}$ containing x and chose affine coordinates t_0, \dots, t_{n+r-1} for U so that x is the origin, and $X \cap U$ is defined by inhomogeneous equations $F_1 = \dots = F_r = 0$. Writing each F_i as a sum of homogeneous terms $F_i^{(j)}$ of degree j ,

$$F_i = \sum_{j=1}^{d_i} F_i^{(j)},$$

W_x is defined by ideal $(\dots F_i^{(j)} \dots)$, $i = 1, \dots, r$, $j = 1, \dots, d_i - 1$. Since we are choosing X to be the generic hypersurface, and as we may also chose x to lie outside any proper closed subset of X , the homogeneous terms $F_i^{(j)} \in K(x)[t_0, \dots, t_{n+r-1}]_j$ will define a complete intersection in $\mathbb{P}_{K(x)}^{n+r-1}$. In particular W_x has codimension $\sum_{i=1}^r (d_i - 1) \leq n + r - 1$ in $\mathbb{P}_{K(x)}^{n+r-1}$, is non-empty (Bezout's theorem!) and has degree $\prod_{i=1}^r (d_i - 1)!$.

Let $W_x^0 \subset W_x$ be the closed subset of lines ℓ containing x with $\ell \subset X$; this is defined by the r additional equations $F_i^{(d_i)} = 0$. Thus, for general (X, x) , W_x^0 has codimension r on W_x (or is empty).

Since $n + r - 1 - \sum_{i=1}^r (d_i - 1) \geq r - 1$, we may intersect W_x with a suitably general linear space $L \subset \mathbb{P}_{K(x)}^{n+r-1}$ to form a closed subscheme $\bar{W}_x \subset W_x$ of dimension $r - 1$ and degree $\prod_{i=1}^r (d_i - 1)!$ and we may chose L with $L \cap W_x^0 = \emptyset$. The cone over \bar{W}_x with vertex x , $\mathcal{C}_x \subset \mathbb{P}_{K(x)}^{n+r}$, is thus a dimension r closed subscheme of degree $\prod_{i=1}^r (d_i - 1)!$ with intersection (set) $\mathcal{C}_x \cap X = \{x\}$. Thus as cycles

$$\mathcal{C}_x \cdot X = \left(\prod_{i=1}^r d_i! \right) \cdot x.$$

Let η be the generic point of X . Taking $x = \eta$ in the above discussion gives

$$\prod_{i=1}^r d_i! \cdot \eta = \mathcal{C}_\eta \cdot X.$$

But \mathcal{C}_η is an r -cycle on $\mathbb{P}_{K(\eta)}^{n+r}$ of degree $\prod_{i=1}^r (d_i - 1)!$, so we have $\mathcal{C}_\eta = \prod_{i=1}^r (d_i - 1)! \cdot L_r$ in $\text{CH}_r(\mathbb{P}_{K(\eta)}^{n+r})$, where $L_r \subset \mathbb{P}_{K(\eta)}^{n+r}$ is any dimension r linear subspace. Since K is infinite, we may choose L_r so that the intersection $L_r \cap X$ has dimension zero. Thus, letting $z = \prod_{i=1}^r (d_i - 1)! \cdot (L_r \cdot X)$, we have

$$\prod_{i=1}^r d_i! \cdot \eta - z_{K(\eta)} = 0$$

in $\mathrm{CH}_0(X_{K(\eta)})$, which gives a decomposition of the diagonal in X of order $\prod_{i=1}^r d_i!$. Thus $\mathrm{Tor}_K(X)$ is finite and divides $\prod_{i=1}^r d_i!$, as desired. \square

Corollary 4.2. *Let $X = X_{d_1, \dots, d_r}^n$ in \mathbb{P}_k^{n+r} be a smooth complete intersection of multi-degree d_1, \dots, d_r and of dimension $n \geq 1$ with $\sum_i d_i \leq n + r$. Then $g\mathrm{Tor}_k(X)$ and $\mathrm{Tor}(X)$ are both finite and both divide $\prod_{i=1}^r d_i!$.*

Proof. Both $g\mathrm{Tor}_k(X)$ and $\mathrm{Tor}(X) := \mathrm{Tor}_{\bar{k}}(X_{\bar{k}})$ divide $\mathrm{Tor}_k(X)$ (Lemma 1.3) so the result follows from Proposition 4.1. \square

5. THE GENERIC CASE

In this section we discuss the case of the generic complete intersection. Let k denote a fixed base-field, for instance the prime field. The bounds we find for the generic case are independent of k , so one could equally well take k to be the reader's favorite field, even an algebraically closed one.

Before going into details, we outline the case of hypersurfaces, which uses all the main ideas.

Let $d!^*$ denote the l.c.m. of the integers $2, \dots, d$. Note that $d!^*$ is inductively the l.c.m. of d and $(d-1)!^*$ (Lemma 5.4). Our main result in the case of hypersurfaces is that the torsion order of level 0 of the generic hypersurface of degree $d \leq n+1$ in \mathbb{P}^{n+1} is divisible by $d!^*$, in other words, if the generic hypersurface admits a decomposition of the diagonal of degree N , then $d!^*$ divides N .

The hypersurfaces of degree $d \leq n+1$ in \mathbb{P}_k^{n+1} are parametrized by a projective space $\mathbb{P}^{N_{n,d}}$ and it is not hard to show that the index over $k(\mathbb{P}^{N_{n,d}})$ of the generic degree d hypersurface X is d . In fact, we have a much stronger statement, namely $\mathrm{CH}_0(X) = \mathbb{Z}$, generated by $X \cdot \ell$ for $\ell \subset \mathbb{P}^{n+1}$ a line (Lemma 5.1(1)).

If we have a decomposition of order N of the diagonal on X ,

$$N \cdot \Delta_X \sim x \times X + \gamma,$$

then since $N = \deg_{k(\mathbb{P}^{N_{n,d}})} x$, it follows that $d|N$. Now degenerate X to the generic degree $d-1$ hypersurface Y in \mathbb{P}^{n+1} plus the hyperplane H given by $x_{n+1} = 0$, and let $Z = Y \cap H$. Here Y and Z are defined over $L := k(\mathbb{P}^{N_{n,d-1}})$. Specializing the above rational equivalence using Lemma 1.7 gives a rational equivalence on $Y \times_L Y$ of the form

$$N \cdot \Delta_Y \sim \bar{x} \times Y + \gamma_1 + \gamma_2$$

with \bar{x} a zero-cycle on Y , γ_1 a dimension n cycle on $Z \times_L Y$ and γ_2 supported in $Y \times D$ for some divisor D on Y . Passing to the generic point of Y , γ_1 gives a 0-cycle on $Z \times_L L(Y)$. The main point is to show that $\mathrm{CH}_0(Z \times_L L(Y))$ is also \mathbb{Z} , generated by intersections from \mathbb{P}^{n-1} (Lemma 5.1(3)), so we can replace γ_1 with $y \times Y + \gamma_3$, where y is a 0-cycle on Z and γ_3 is supported on $Z \times D'$ for some divisor D' on Y (Lemma 5.2). In other words,

$$N \cdot \Delta_Y \sim (\bar{x} + y) \times Y + \gamma_2 + \gamma_3,$$

so Y admits a decomposition of the diagonal of degree N . Now use induction on d to conclude that $(d-1)!^*|N$. As we already know that $d|N$, we find $d!^*|N$.

Now for the details. Fix integers $n, r \geq 1$. For an integer d , let $\mathcal{S}_{d,n+r}$ be the set of indices $I = (i_0, \dots, i_{n+r})$ with $0 \leq i_j$ and $\sum_j i_j = d$. We let $\mathcal{S}_i = \mathcal{S}_{d_i, n+r}$ and let $N_i := \#\mathcal{S}_i$. Let $\{u_i^{(I)} | I \in \mathcal{S}_i\}$ be homogeneous coordinates for \mathbb{P}^{N_i} and let x_0, \dots, x_{n+r} be homogeneous coordinates for \mathbb{P}^{n+r} . The universal family of intersections of multi-degree d_1, \dots, d_r in \mathbb{P}^{n+r} , $\mathcal{X}^{d_*, n}$, is the subscheme of $\mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r} \times \mathbb{P}^{n+r}$ defined by the multi-homogeneous ideal in the polynomial ring $k[\{u_i^{(I)}\}_{I \in \mathcal{S}_i, i=1, \dots, r}, x_0, \dots, x_{n+r}]$ generated by the elements

$$\sum_{I \in \mathcal{S}_i} u_i^{(I)} x^I; \quad i = 1, \dots, r$$

where as usual $x^I = x_0^{i_0} \dots x_{n+r}^{i_{n+r}}$ for $I = (i_0, \dots, i_{n+r})$. We let $\eta := \eta_{d_*, n}$ denote the generic point of $\mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r}$ and let $\mathcal{X}_\eta^{d_*, n}$ denote the fiber product

$$\mathcal{X}_\eta^{d_*, n} := \mathcal{X}^{d_*, n} \times_{\mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r}} \eta \subset \mathbb{P}_\eta^{n+r}.$$

By Proposition 4.1, we know that if $\sum_{i=1}^r d_i \leq n+r$, then $\text{Tor}_{k(\eta)}(\mathcal{X}_\eta^{d_*, n})$ is finite and divides $\prod_i d_i!$. We turn to a computation of a lower bound.

Let $H \subset \mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r} \times \mathbb{P}^{n+r}$ be the subscheme defined by $(x_{n+r} = 0)$, let $\mathcal{X}_H^{d_*, n} := \mathcal{X}^{d_*, n} \cap H$ and let $\mathcal{X}_{H, \eta}^{d_*, n} := \mathcal{X}_H^{d_*, n} \cap \eta$. Let $\eta' := \eta_{d_*, n-1}$.

We separate the indices \mathcal{S}_i into two disjoint subsets \mathcal{S}_i^0 and \mathcal{S}_i^1 , with \mathcal{S}_i^0 the set of (i_0, \dots, i_{n+r}) with $i_{n+r} = 0$ and \mathcal{S}_i^1 those with $i_{n+r} > 0$. We set $v_i^{(I)} = u_i^{(I)}$ for $I \in \mathcal{S}_i^0$ and $w_i^{(I)} = u_i^{(I)}$ for $I \in \mathcal{S}_i^1$. We write $k(\{u_i^{(I)}\}_0)$ for the field extension of k generated by the ratios $u_i^{(I)}/u_i^{(I')}$ $I \neq I'$, and similarly for $k(\{v_i^{(I)}\}_0)$, giving us the field extension $k(\{v_i^{(I)}\}_0) \subset k(\{u_i^{(I)}\}_0)$. We note that $k(\{u_i^{(I)}\}_0) = k(\eta)$, $k(\{v_i^{(I)}\}_0) = k(\eta')$ and the $k(\eta)$ -scheme $\mathcal{X}_{H, \eta}^{d_*, n}$ is canonically isomorphic to the base-change of the $k(\eta')$ -scheme $\mathcal{X}_{\eta'}^{d_*, n-1}$ via the base extension $k(\eta') \subset k(\eta)$:

$$\mathcal{X}_{H, \eta}^{d_*, n} \cong \mathcal{X}_{\eta'}^{d_*, n-1} \otimes_{k(\eta')} k(\eta).$$

This defines for us the projection $q_1 : \mathcal{X}_{H, \eta}^{d_*, n} \rightarrow \mathcal{X}_{\eta'}^{d_*, n-1}$.

Let $K = k(\eta)(\mathcal{X}_\eta^{d_*, n}) = k(\mathcal{X}^{d_*, n})$. We have the morphism of $k(\eta')$ -schemes

$$\pi : \mathcal{X}_{H, \eta}^{d_*, n} \otimes_{k(\eta_{d_*, n})} K \rightarrow \mathcal{X}_{\eta'}^{d_*, n-1}$$

formed by the composition

$$\mathcal{X}_{H, \eta}^{d_*, n} \otimes_{k(\eta)} K \xrightarrow{p_1} \mathcal{X}_{H, \eta}^{d_*, n} \xrightarrow{q_1} \mathcal{X}_{\eta'}^{d_*, n-1}$$

Lemma 5.1. *1. For $i = 0, \dots, n$, the intersection map*

$$\text{CH}_{r+i}(\mathbb{P}_{k(\eta)}^{n+r}) \rightarrow \text{CH}_i(\mathcal{X}_\eta^{d_*, n})$$

is an isomorphism.

2. For $i = 0, \dots, n-1$, the pullback

$$\pi^* : \mathrm{CH}_i(\mathcal{X}_{\eta'}^{d_*, n-1}) \rightarrow \mathrm{CH}_i(\mathcal{X}_{H, \eta}^{d_*, n} \otimes_{k(\eta)} K)$$

is an isomorphism.

3. For $i = 0, \dots, n-1$, the intersection map

$$\mathrm{CH}_{r+i}(\mathbb{P}_K^{n+r}) \rightarrow \mathrm{CH}_i(\mathcal{X}_{H, \eta}^{d_*, n} \otimes_{k(\eta)} K)$$

is an isomorphism.

Proof. Noting that the base-extension $\mathrm{CH}_*(\mathbb{P}_{k(\eta)}^{n+r}) \rightarrow \mathrm{CH}_*(\mathbb{P}_K^{n+r})$ is an isomorphism, the assertion (3) follows from (1) (for $n-1$) and (2). For (1), the projection

$$p_2 : \mathcal{X}^{d_*, n} \rightarrow \mathbb{P}^{n+r}$$

expresses $\mathcal{X}^{d_*, n}$ as $\mathbb{P}^{N_1-1} \times \dots \times \mathbb{P}^{N_r-1}$ -bundle over \mathbb{P}^{n+r} , with fibers embedded in $\mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r}$ linearly in each factor. Thus $\mathrm{CH}_*(\mathcal{X}^{d_*, n})$ is generated by $\mathrm{CH}_*(\mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r} \times \mathbb{P}^{n+r})$ via restriction. After localization at η , this shows that $\mathrm{CH}_*(\mathcal{X}_{\eta}^{d_*, n})$ is generated by $\mathrm{CH}_*(\mathbb{P}_{k(\eta)}^{n+r})$ via restriction. The fact that the surjective map $\mathrm{CH}_{r+i}(\mathbb{P}_{k(\eta)}^{n+r}) \rightarrow \mathrm{CH}_i(\mathcal{X}_{\eta}^{d_*, n})$ is also injective in the stated range follows by noting that the intersection pairing on $\mathcal{X}_{\eta}^{d_*, n}$ is non-degenerate when restricted to these cycles. This proves (1).

For (2), fix for each i the index $I_i^0 := (d_i, 0, \dots, 0)$, and the index $I_i^1 := (0, \dots, 0, d_i)$, and for each homogeneous variable $w_i^{(I)}$, let $w_i^{(I)0}$ be the corresponding affine coordinate $w_i^{(I)}/v_i^{(I^0)}$. Similarly, we let $v_i^{(I)0} = v_i^{(I)}/v_i^{(I^0)}$. Let $y_i = x_i/x_0$, $i = 1, \dots, n+r$, $y_0 = 1$. The field extension $k(\eta') \rightarrow K$ is isomorphic to the field extension given by including the constants $k(\{v_i^{(I)}\}_0)$ of the $k(\{v_i^{(I)}\}_0)$ -algebra A ,

$$A := k(\{v_i^{(I)}\}_0, y_1, \dots, y_{n+r})[\{w_i^{(I)0}\}]/(\dots, \sum_{I \in \mathcal{S}_i^0} v_i^{(I)0} \cdot y^I + \sum_{I' \in \mathcal{S}_i^1} w_i^{(I')0} \cdot y^{I'}, \dots)$$

into the quotient field L of A . In each defining relation for A , we can solve for $w_i^{(I^1)0}$ in terms of the y_i 's and the other $w_i^{(I')0}$'s. After eliminating each $w_i^{(I^1)0}$ in this way, we see that A is a polynomial algebra over $k(\{v_i^{(I)}\}_0, y_1, \dots, y_{n+r})$. The y_i and the $w_i^{(I')0}$, after removing $w_i^{(I^1)0}$ for each i , therefore form an algebraically independent set of generators for L over $k(\{v_i^{(I)}\}_0)$, and thus K is a pure transcendental extension of $k(\eta')$. As Chow groups are invariant under base-change by purely transcendental field extensions, this proves (2). \square

Lemma 5.2. *Take γ in $\mathrm{CH}_n(\mathcal{X}_{H, \eta}^{d_*, n} \times_{k(\eta)} \mathcal{X}_{\eta}^{d_*, n})$. Then there is a zero cycle y on $\mathcal{X}_{H, \eta}^{d_*, n}$ a proper closed subset D' of $\mathcal{X}_{\eta}^{d_*, n}$ and a cycle γ' supported on*

$\mathcal{X}_{H,\eta}^{d_*,n} \times_{k(\eta)} D'$ such that

$$\gamma = y \times \mathcal{X}_{\eta}^{d_*,n} + \gamma'$$

in $\text{CH}_n(\mathcal{X}_{H,\eta}^{d_*,n} \times_{k(\eta)} \mathcal{X}_{\eta}^{d_*,n})$. Furthermore the degree of y is divisible by $\prod_{i=1}^r d_i$.

Proof. Let ξ denote the generic point of $\mathcal{X}_{\eta}^{d_*,n}$. By Lemma 5.1(3), the class of the restriction $j^*\gamma$ of γ to $\mathcal{X}_{H,\eta}^{d_*,n} \times_{k(\eta)} \xi$ is of the form

$$j^*\gamma = M \cdot L \cdot \mathcal{X}_{H,\eta}^{d_*,n} \times_{k(\eta)} \xi,$$

where L is a linear subspace of $H \subset \mathbb{P}^{n+r}$, M an integer. Letting $y \in \text{CH}_0(\mathcal{X}_{H,\eta}^{d_*,n})$ be the 0-cycle $M \cdot L \cdot \mathcal{X}_{H,\eta}^{d_*,n}$, the result follows from the localization theorem for the Chow groups; the assertion on the degree follows from the fact that $\mathcal{X}_{H,\eta}^{d_*,n}$ has degree $\prod_{i=1}^r d_i$ and hence y has degree $M \cdot \prod_{i=1}^r d_i$. \square

Definition 5.3. For a natural number $n \geq 1$, we let $n!^*$ denote the least common multiple of the numbers $1, 2, \dots, n$.

Lemma 5.4. Let d_1, \dots, d_r be a sequence of positive natural numbers. Then the product $\prod_{i=1}^r (d_i!^*)$ is equal to the least common multiple M of all products $i_1 \cdot \dots \cdot i_r$ with $1 \leq i_j \leq d_j$, $j = 1, \dots, r$.

Proof. Fix a prime number p . For each $j = 1, \dots, r$, let i_j^* be an integer with $1 \leq i_j^* \leq d_j$ and with p -adic valuation $\nu_p(i_j^*)$ equal to $\nu_p(d_j!^*)$. Then

$$\nu_p\left(\prod_{j=1}^r i_j^*\right) = \nu_p\left(\prod_{i=1}^r (d_i!^*)\right)$$

and $\nu_p(\prod_{j=1}^r i_j) \leq \nu_p(\prod_{j=1}^r i_j^*)$ for all sequences i_1, \dots, i_r with $1 \leq i_j \leq d_j$. Thus $\nu_p(M) = \nu_p(\prod_{i=1}^r i_j^*) = \nu_p(\prod_{i=1}^r (d_i!^*))$. Since p was arbitrary, this gives $M = \prod_{i=1}^r (d_i!^*)$. \square

Theorem 5.5. For integers d_1, \dots, d_r with $\sum_i d_i \leq n+r$, $\prod_{i=1}^r d_i!^*$ divides $\text{Tor}_{k(\eta)}(\mathcal{X}_{\eta}^{d_*,n})$.

Proof. We may suppose that $d_1 > 1$. Let $d'_* = (d_1 - 1, d_2, \dots, d_r)$. Let \mathcal{O} be the local ring of the origin in $\mathbb{A}_{k(\eta)}^1 = \text{Spec } k(\eta)[t]$ and let $\tilde{\mathcal{X}}$ be the subscheme of $\mathbb{P}^{N_1} \times \dots \times \mathbb{P}_{\mathcal{O}}^{N_r}$ defined by the homogeneous ideal (f_1, \dots, f_r) , with

$$f_j = \begin{cases} \sum_{I \in \mathcal{S}_{d_j, n+r}} u_j^{(I)} x^I & \text{for } j \neq 1 \\ t \cdot \sum_{I \in \mathcal{S}_{d_1, n+r}} u_1^{(I)} x^I + (1-t) \cdot x_{n+r} \cdot \sum_{J \in \mathcal{S}_{d_1-1, n+r}} u_1^{(J)} x^J & \text{for } j = 1. \end{cases}$$

The generic fiber of $\tilde{\mathcal{X}}$ is thus isomorphic to $\mathcal{X}_{\eta}^{d_*,n} \times_{k(\eta)} k(\eta, t)$ and the special fiber is $\mathcal{X}_{\eta}^{d'_*,n} \cup H$.

Suppose that $\mathcal{X}_\eta^{d_*,n}$ admits a decomposition of the diagonal of order N :

$$N \cdot \Delta_{\mathcal{X}_\eta^{d_*,n}} = x \times \mathcal{X}_\eta^{d_*,n} + \gamma$$

with γ supported on $\mathcal{X}_\eta^{d_*,n} \times D$ for some divisor D . By Lemma 5.1, $\deg x$ is divisible by $\prod_{i=1}^r d_i$, and thus $\prod_{i=1}^r d_i$ divides N .

By applying Totaro's specialization lemma (Lemma 1.7) to the family $\mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}$, the diagonal for $\mathcal{X}_\eta^{d'_*,n}$ admits a decomposition of the form

$$N \cdot \Delta_{\mathcal{X}_\eta^{d'_*,n}} = \bar{x} \times \mathcal{X}_\eta^{d'_*,n} + \gamma_1 + \gamma_2$$

with γ_1 supported in $\mathcal{X}_{H,\eta}^{d'_*,n} \times \mathcal{X}_\eta^{d'_*,n}$ and γ_2 supported in $\mathcal{X}_\eta^{d'_*,n} \times D_2$ for some divisor D_2 on $\mathcal{X}_\eta^{d'_*,n}$. By Lemma 5.2, we have the identity

$$\gamma_1 = y \times \mathcal{X}_\eta^{d'_*,n} + \gamma_3$$

with y a zero-cycle on $\mathcal{X}_\eta^{d'_*,n}$ and γ_3 supported on $\mathcal{X}_\eta^{d'_*,n} \times D_3$ for some divisor D_3 . Thus, the diagonal on $\mathcal{X}_\eta^{d'_*,n}$ admits a decomposition of order N as well. By induction $(d_1 - 1)! \cdot \prod_{i=2}^r (d_i!)$ divides N ; by symmetry $(d_j - 1)! \cdot \prod_{i=1, i \neq j}^r (d_i!)$ divides N for all j with $d_j > 1$. As we have already seen that $\prod_i d_i$ divides N , Lemma 5.4 completes the proof. \square

We also have a lower bound for the generic complete intersection with a rational point.

Corollary 5.6. *For integers d_1, \dots, d_r with $\sum_i d_i \leq n + r$, let K be the function field of the generic complete intersection of multi-degree d_1, \dots, d_r , $K := k(\eta)(\mathcal{X}_\eta^{d_*,n})$. Then $(1/\prod_{i=1}^r d_i) \prod_{i=1}^r (d_i!)$ divides $\operatorname{Tor}_K(\mathcal{X}_\eta^{d_*,n} \times_{k(\eta)} K)$.*

Proof. Let $X = \mathcal{X}_\eta^{d_*,n}$. By Lemma 5.1, $I_X = \prod_{i=1}^r d_i$ and thus by Lemma 1.10, $\operatorname{Tor}_{k(\eta)}(\mathcal{X}_\eta^{d_*,n})$ divides $I_X \cdot \operatorname{Tor}_K(\mathcal{X}_\eta^{d_*,n} \times_{k(\eta)} K)$. Clearly $\prod_{i=1}^r d_i$ divides $\prod_{i=1}^r (d_i!)$, whence the result. \square

Example 5.7 (Generic cubic hypersurfaces). For the generic cubic hypersurface $X := \mathcal{X}_\eta^{3,n}$, $n \geq 2$, we thus have $\operatorname{Tor}_{k(\eta)} X = 6$ and the generic cubic hypersurface with a rational point X_K , $K = k(\eta)(X)$, has $2 \mid \operatorname{Tor}_K X_K \mid 6$. If X_K were to admit a dominant rational map $\mathbb{P}^n \dashrightarrow X_K$ of degree prime to 3, then by Proposition 2.7(2), we would have $\operatorname{Tor}_K X_K = 2$. We know that if a cubic hypersurface X has a line (defined over the base-field) then there is a degree two dominant rational map $\mathbb{P}^n \dashrightarrow X$ (see for example [23, §5]), but it is not clear if this is the case if we only assume that X has a (suitably general) rational point.

However, as pointed out by a referee, the generic cubic *surface* with a rational point does have $\operatorname{Tor}_K X_K = 6$, at least if k has characteristic not equal to 3. Indeed, if we take a field k_0 of characteristic different from three, containing a primitive cube root of 1, and let k be a pure transcendental

extension of k_0 , we may find an element $a \in k$ that is not a cube. Then the smooth cubic surface $Y \subset \mathbb{P}_k^3$ given by $x^3 + y^3 + z^3 + at^3 = 0$ has a rational point but also has $Br(Y)/Br(k) \cong (\mathbb{Z}/3)^2$ (see for example [20]), and thus $\text{Tor}_k(Y)$ is divisible by 3. Specializing the generic cubic surface with a rational point X_K to Y , we may apply the divisibility lemma 1.5 to conclude that $3 | \text{Tor}_K X_K$. In particular, the generic cubic surface with a rational point does not admit a rational map $\mathbb{P}^2 \dashrightarrow X_K$ of degree not divisible by 6.

Example 5.8 (Generic cubic hypersurfaces with a line). Take $n \geq 2$. For X a cubic hypersurface in \mathbb{P}_L^{n+1} (defined over some field $L \supset k$), we have the Fano variety of lines on X , F_X , a closed subscheme of the Grassmann variety $\text{Gr}(2, n+2)_L$. In fact, if $U \rightarrow \text{Gr}(2, n+2)$ is the universal rank two bundle, and f is the defining equation for X , then F_X is the closed subscheme defined by the vanishing of the section of the rank four bundle $\text{Sym}^3 U$ determined by f . In particular, the class of F_X in $\text{CH}^4(\text{Gr}(2, n+2)_L)$ is given by the Chern class $c_4(\text{Sym}^3 U)$. One computes this easily as $c_4 = 9c_2^2(U) + 18c_1(U)^2c_2(U)$. As $c_2(U)^n$ and $c_2(U)^{n-2}c_1(U)^2$ both have degree one, we see that $F_X \cdot c_2(U)^{n-2}$ has degree 27, and thus I_{F_X} divides 27. This 27 is of course the famous 27 lines on a cubic surface, as intersecting F_X with $c_2(U)^{n-2}$ in $\text{Gr}(2, n+2)$ is the same as taking the Fano variety of the intersection of X with a general \mathbb{P}^3 in \mathbb{P}^{n+1} . See for example [11, 14.7.13] for details of the Chern class computation.

Taking $X = \mathcal{X}_\eta^{3,n}$, and letting $K = k(\eta)(F_X)$, it follows from Lemma 1.10 that $6 = \text{Tor}_{k(\eta)}(\mathcal{X}_\eta^{d_*,n})$ divides $27 \cdot \text{Tor}_K(\mathcal{X}_\eta^{d_*,n} \times_{k(\eta)} K)$; since we have the degree two rational map $\mathbb{P}_K^n \dashrightarrow \mathcal{X}_\eta^{d_*,n} \times_{k(\eta)} K$, we have $\text{Tor}_K(\mathcal{X}_\eta^{d_*,n} \times_{k(\eta)} K) = 2$. In particular, the generic cubic with a line is not stably rational over its natural field of definition $k(\eta)(F_X)$.

We are indebted to J.-L. Colliot-Thélène for the next example (see [8, Théorème]), which improves the bounds and simplifies the argument of an example in an earlier version of this paper.

Example 5.9 (Cubics over a “small” field). Take $n \geq 2$. We consider a DVR \mathcal{O} with quotient field K and residue field k (of characteristic $\neq 2$), and a degree 3 hypersurface $\mathcal{X} \subset \mathbb{P}_{\mathcal{O}}^{n+1}$. Let $X = \mathcal{X}_K$ and $Y = \mathcal{X}_k$. We suppose that X is smooth and $Y = Q \cup H$, with Q a smooth quadric and H a hyperplane. Furthermore, we assume

- (1) $I_Q = 1$.
- (2) Q and H intersect transversely.
- (3) $I_{Q \cap H} = 2$.

From Proposition 4.1, we know that $\text{Tor}_K(X)$ is finite and divides 6. We will show that 2 divides $\text{Tor}_K(X)$.

For this, suppose we have a decomposition of the diagonal of X of order N . We note that our family \mathcal{X} satisfies the hypotheses of Lemma 1.8, with

$Y_1 = Q$, $Y_2 = H$, and $r = 1$. By Remark 1.9, $N \cdot (\text{CH}_0(Q)/i_{Q \cap H*}(\text{CH}_0(Q \cap H))) = 0$; considering degrees, we see that $2|N$.

To construct an explicit example, recall [19] that a field k has *u-invariant* $u(k) \geq r$ if there exists an anisotropic quadratic form over k of dimension r . The above construction gives us a cubic hypersurface X of dimension $n \geq 2$ over $K := k((x))$ with $2|\text{Tor}_K(X)$ and $X(K) \neq \emptyset$ if k is an infinite field of characteristic $\neq 2$ with $u\text{-invariant} \geq n + 1$. Indeed, take a anisotropic quadratic form q_0 in $n + 1$ -variables X_0, \dots, X_n , choose $\alpha \in k^\times$ represented by q_0 and let $q = q_0 - \alpha \cdot X_{n+1}^2$, so q is non-degenerate. Let $Q \subset \mathbb{P}_k^{n+1}$ be the quadric defined by q and let H be the hyperplane $X_{n+1} = 0$. Take a cubic form $c_0 \in k[X_0, \dots, X_{n+1}]$ and let $c = xc_0 + q \cdot X_{n+1} \in k[[x]][X_0, \dots, X_{n+1}]$. Since k is infinite, we can choose c_0 so that the subscheme X of \mathbb{P}_k^{n+1} defined by c is smooth (and hence geometrically integral); it suffices to choose c_0 so that $c_0 = 0$ is smooth and intersects Q and H transversely. Clearly $I_Q = 1$, Q and H intersect transversely and $I_{Q \cap H} = 2$, giving us the desired example.

For example, \mathbb{F}_p has $u\text{-invariant}$ 2, and \mathbb{Q}_p and $\mathbb{F}_p((t))$ both have $u\text{-invariant}$ 4 (see for example [19]). Thus there are cubic threefolds X over $K := \mathbb{Q}_p((x))$ with $2|\text{Tor}_K(X)$ and with $X(K) \neq \emptyset$. Similarly, there are examples of such cubic threefolds over $K = \mathbb{F}_p((t))((x))$ for $p \neq 2$. Over $K = \mathbb{Q}((x))$ or even over $K = \mathbb{R}((x))$ there are cubic hypersurfaces X of dimension n over K for arbitrary $n \geq 2$, with $2|\text{Tor}_K(X)$ and $X(K) \neq \emptyset$. As in the previous example, we may pass to an odd degree field extension L of K to find a cubic hypersurface X_L with a line, and with $\text{Tor}_L(X_L) = 2$; all these cubics are thus not stably rational over their corresponding field of definition.

Remark 5.10. As mentioned in the introduction, Colliot-Thélène and Pirutka have constructed cubic threefolds over a p -adic field [6, Théorème 1.21] and over $\mathbb{F}_p((x))$ [6, Remarque 1.23] with non-zero torsion order and having a rational point.

6. TORSION ORDER FOR COMPLETE INTERSECTIONS IN A PROJECTIVE SPACE: A LOWER BOUND

As in the previous sections, we consider smooth complete intersection subschemes X of \mathbb{P}^{n+r} of multi-degree d_1, \dots, d_r .

By saying a property holds for a very general complete intersection in \mathbb{P}_k^{n+r} of multi-degree d_1, \dots, d_r we mean that there is a countable union F of proper closed subsets of the parameter scheme of such complete intersections (an open in a product of projective spaces over k) such that the property holds for X_b if $b \notin F$.

Recall that for X a proper, separable L -scheme for some field L , and \bar{L} the algebraic closure of L , we have defined $\text{Tor}^{(i)}(X) := \text{Tor}_{\bar{L}}^{(i)}(X_{\bar{L}})$.

Theorem 6.1. *Let k be a field of characteristic zero. Let d_1, \dots, d_r and $n \geq 3$ be integers with $d' := \sum_{j=1}^r d_j \leq n + r$. Let p be a prime number.*

Suppose that

$$(6.1) \quad d_i \geq p \cdot \left\lceil \frac{n + r + 1 - d' + d_i}{p + 1} \right\rceil$$

for some i , $1 \leq i \leq r$. Then $p | \text{Tor}^{(n-2)}(X)$ for all very general $X = X_{d_1, \dots, d_r} \subset \mathbb{P}_k^{n+r}$.

Corollary 6.2. *Let k , d_1, \dots, d_r , n and p be as in Theorem 6.1 and suppose that d_i satisfies (6.1). Then $p | \text{Tor}(X)$ for all very general $X = X_{d_1, \dots, d_r} \subset \mathbb{P}_k^{n+r}$.*

Proof. $\text{Tor}^{(n-2)}(X)$ divides $\text{Tor}(X) := \text{Tor}^{(0)}(X)$ by Lemma 1.3(1). \square

Remarks 6.3. 1. We know that $\text{Tor}(X)$ is finite for all $X = X_{d_1, \dots, d_r} \subset \mathbb{P}^{n+r}$ with $\sum_j d_j \leq n + r$ by Proposition 4.1 and hence $\text{Tor}^{(n-2)}(X)$ is also finite.

2. For $p = 2$ and for hypersurfaces, the corollary follows directly from the results in Totaro's paper [28].

3. We only use the hypothesis of characteristic zero to allow for a specialization to characteristic p , where p is the prime number in the statement. For k a field of positive characteristic, the analogous result holds, but only for $p = \text{char } k$.

4. There are two interesting cases of complete intersection threefolds we would like to mention: that of a multi-degree $(3, 2)$ complete intersection in \mathbb{P}^5 and a multi-degree $(2, 2, 2)$ complete intersection in \mathbb{P}^6 (see the recent results of Hassett-Tschinkel [13]). In both cases we take $d_i = 2$ and get a divisibility by 2. Notice that in the $(2, 3)$ case taking $d_i = 3$ and $p = 3$ works.

Proof of the theorem. This is another application of the argument of Kollár [18], as used for example by Totaro [28], Colliot-Thélène and Pirutka [7], or Okada [24]. We may reorder the d_j so that $d_i = d_1$. We first assume that p divides d_1 , $d_1 = q \cdot p$. Take f and g suitably general homogeneous polynomials of degree d_1 and q , respectively, and let f_2, \dots, f_r be suitably general homogeneous polynomials, with f_j of degree d_j , $j = 2, \dots, r$. We take these to be in the polynomial ring $\mathcal{O}[X_0, \dots, X_{n+r}]$, where \mathcal{O} is a complete (hence excellent) discrete valuation ring with maximal ideal (t) , residue field $k = \bar{\mathbb{F}}_p$, the algebraic closure of \mathbb{F}_p , and with quotient field K a field of characteristic zero. We let $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}$ be the closed subscheme of a weighted projective space $\mathbb{P} = \text{Proj } \mathcal{O}[X_0, \dots, X_{n+r}, Y]$, with the X_i having weight 1 and Y having weight q , defined by the homogeneous ideal

$$(f_2, \dots, f_r, Y^p - f, g - tY).$$

The generic fiber $X := \mathcal{X}_K$ is isomorphic to the complete intersection subscheme of \mathbb{P}_K^{n+r} defined by $g^p - t^p f = f_2 = \dots = f_r = 0$ and the special fiber $Y := \mathcal{X}_k$ is the cyclic p to 1 cover $Y \rightarrow W$, with $W \subset \mathbb{P}_k^{n+r}$ the complete intersection defined by $\bar{g} = \bar{f}_2 = \dots = \bar{f}_r = 0$, and $y^p = f|_W$.

For general f, g, f_2, \dots, f_r , X and W are smooth, and Y has only finitely many singularities, which may be resolved by an explicit iterated blow-up $q : Z \rightarrow Y$ which is totally CH_0 -trivial: for details, see Proposition 7.5 if $p \geq 3$. If $p = d_1 = 2$, then we use Lemma 7.7 and Proposition 7.8 for the construction of the resolution of singularities and the proof that the resolution morphism q is totally CH_0 -trivial. Kollár shows in addition, that under the assumption (6.1), one has $H^0(Z, \Omega_{Z/k}^{n-1}) \neq \{0\}$. In somewhat more detail, Kollár (see [18, §15, Lemma 16] defines an invertible sheaf Q (denoted $\pi^*Q(L, s)$ in *loc. cit.*) with an injection $Q \rightarrow (\Omega_{Y/k}^{n-1})^{**}$, where ** denotes the double dual. A local computation (see [7], [24] or Remark 7.18 for details) in a neighborhood of the finitely many singularities of Y shows that this injection extends to an injection $q^*Q \rightarrow \Omega_{Z/k}^{n-1}$; here is where the condition $n \geq 3$ is used. In addition, q^*Q is isomorphic to the pullback to Z of $\omega_W \otimes \mathcal{O}_W(d_1)$, where ω_W is the canonical sheaf on W . As $\omega_W = \mathcal{O}_W(d_1/p + \sum_{j \geq 2} d_j - n - r - 1)$, we have a non-zero section of $\Omega_{Z/k}^{n-1}$ if $d_1(p+1)/p \geq n + r + 1 - \sum_{i=2}^r d_i$, which is exactly the condition in the statement of the theorem.

By Proposition 4.1, we know that $\text{Tor}(X_{\bar{K}})$ is finite and thus $\text{Tor}^{(n-2)}(X_{\bar{K}})$ is finite as well. The specialization result Proposition 3.2 thus implies that $\text{Tor}^{(n-2)}(Z_{\bar{k}})$ is finite and divides $\text{Tor}^{(n-2)}(X_{\bar{K}})$. By [12, Prop. 4.2.33], [5, Thm. 3.1.8], and [10, III.3.Prop. 4], correspondences on $Z \times_k Z$ act on $H^0(Z, \Omega_{Z/k}^{n-1})$ and if γ is a correspondence on $Z \times_k Z$ supported in some $Z' \times Z$ with $\dim_k Z' \leq n-2$, then by [5, Proposition 3.2.2(2)], γ_* acts by zero on $H^0(Z, \Omega_{Z/k}^{n-1})$. Similarly, if γ is a correspondence on $Z \times Z$, supported in $Z \times D$ for some divisor $D \subset Z$, then $\gamma_*(\omega)|_{Z \setminus D} = 0$ for each $\omega \in H^0(Z, \Omega_{Z/k}^{n-1})$; as $\Omega_{Z/k}^{n-1}$ is locally free, it follows that $\gamma_*(\omega) = 0$. Thus, if Δ_Z admits a decomposition of order N and level $n-2$, this implies that $N \cdot \omega = 0$ for all $\omega \in H^0(Z, \Omega_{Z/k}^{n-1})$, and since $H^0(Z, \Omega_{Z/k}^{n-1})$ is a non-zero k -vector space, this implies that $p|N$. Since $\text{Tor}^{(n-2)}(Z_{\bar{k}})$ divides $\text{Tor}^{(n-2)}(X_{\bar{K}})$, it follows that $p|\text{Tor}^{(n-2)}(X_{\bar{K}})$ and Corollary 1.6 finishes the proof in this case.

In the case of a general d_1 , write $d_1 = q \cdot p + c$, $0 < c < p$, and consider a family $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}$ defined by a homogeneous ideal of the form

$$(f_2, \dots, f_r, (Y^p - h)s + tu, g - tY),$$

with $u, h, g, s \in \mathcal{O}[X_0, \dots, X_{n+r}]$, u of degree d_1 , h of degree pq , g of degree q and s of degree c , suitably general, and with Y as above of weight q . The generic fiber X is the complete intersection $f_1 = f_2 = \dots = f_r = 0$, with $f_1 = (g^p - t^p h)s + t^{p+1}u$; the special fiber Y has two components Y_1, Y_2 , with Y_1 the p to 1 cyclic cover of $W := (\bar{f}_2 = \dots = \bar{f}_r = \bar{g} = 0)$, branched along $W \cap (h = 0)$. We take $q : Z \rightarrow Y_1$ the resolution as in the previous case. Having chosen h, g, s , we may take u sufficiently general so that X is a smooth complete intersection.

Since \mathcal{O} is excellent, we are free to make a finite extension L of K , take the integral closure \mathcal{O}_L of \mathcal{O} in L , replace \mathcal{O} with the localization \mathcal{O}' at a maximal ideal of \mathcal{O}_L , and replace \mathcal{X} with $\mathcal{X} \otimes_{\mathcal{O}} \mathcal{O}'$; changing notation, we may assume that $\mathrm{Tor}_K^{(n-2)}(X)$ is the geometric torsion order $\mathrm{Tor}^{(n-2)}(X)$. By Proposition 3.3, the smooth proper k -scheme Z admits a decomposition of the diagonal as

$$N \cdot \Delta_Z = \gamma + \gamma_1 + \gamma_2,$$

with $N = \mathrm{Tor}^{(n-2)}(X)$, γ supported in $Z_{n-2} \times Z$ with $\dim Z_{n-2} \leq n-2$, γ_1 supported in $q^{-1}(Y_1 \cap Y_2) \times Z$ and γ_2 supported in $Z \times D$ for some divisor D on Z .

We may take the degree c part s as general as we like. In particular, we may assume that $Y_1 \cap Y_2$ is contained in the smooth locus of Y_1 and is thus isomorphic to a closed subscheme Z' of Z .

Our decomposition of the diagonal on Z gives the relation

$$N \cdot \omega = \gamma_{1*}\omega$$

for each $\omega \in H^0(Z, \Omega_Z^{n-1})$. Indeed,

$$N \cdot \omega = N \cdot \Delta_{Z*}\omega = \gamma_{1*}\omega + \gamma_{2*}\omega + \gamma_*\omega.$$

But γ_* factors through the restriction to Z_{n-2} , so $\gamma_*\omega = 0$. Similarly, $\gamma_{2*}\omega$ is a global section of Ω_Z^{n-1} supported in D , which is zero, since Ω_Z^{n-1} is a locally free sheaf.

One computes that the canonical class of $Y_1 \cap Y_2$ is anti-ample and thus the canonical line bundle on the dimension $n-1$ subscheme Z' has no sections. Note that Z' is a cyclic p to 1 cover of the complete intersection $W \cap V(\bar{s})$. If s is general then there is a rational resolution of singularities \tilde{Z}' (Proposition 7.8, Lemma 7.9), hence the canonical line bundle of \tilde{Z}' has no non-vanishing sections. But $\gamma_{1*}\omega$ factors through the restriction of ω to \tilde{Z}' , hence $\gamma_{1*}\omega = 0$. Since h has degree $q \cdot p$ in the range needed to give the existence of a non-zero ω in $H^0(Z, \Omega_Z^{n-1})$, we conclude as before that $p|N$. \square

Example 6.4. We consider the case of hypersurfaces of degree d in \mathbb{P}^{n+1} , $n \geq 3$. The theorem says that p divides $\mathrm{Tor}^{(n-2)}(X)$ for very general degree $d \leq n+1$ hypersurfaces X in \mathbb{P}^{n+1} if

$$d \geq p \cdot \left\lceil \frac{n+2}{p+1} \right\rceil$$

For $p=2$, this is the range considered by Totaro; for $p=3$, the first case is degree 6 in \mathbb{P}^6 . For the extreme case of degree $d = n+1$ in \mathbb{P}^{n+1} , we have $p|\mathrm{Tor}^{(n-2)}(X)$ for all p dividing $n+1$.

7. AN IMPROVED LOWER BOUND FOR THE VERY GENERAL COMPLETE INTERSECTION

In this section we extend Theorem 6.1 to cover prime powers. The basic idea is to replace the differential forms with Hodge-Witt cohomology. We

are grateful to the referee for providing the argument for the next lemma which shows that a cycle on $Z \times Z$, supported on $Z' \times Z$ with $\dim Z' \leq n-2$, acts trivially on $H^0(Z, W_m \Omega_Z^{n-1})$.

Lemma 7.1. *Let k be a perfect field of positive characteristic p , and X, Y smooth, equidimensional, and quasi-projective k -schemes. Set $n = \dim X$ and $\mathrm{CH}_{\mathrm{prop}/Y}^n(X \times Y) = \varinjlim_Z \mathrm{CH}_{\dim Y}^n(Z)$, where the limit is over all closed subsets $Z \subset X \times Y$ that are proper over Y . For $\alpha \in \mathrm{CH}_{\mathrm{prop}/Y}^n(X \times Y)$ denote by*

$$\alpha_* : \oplus_{i,j} H^i(X, W_m \Omega^j) \rightarrow \oplus_{i,j} H^i(Y, W_m \Omega^j)$$

the map induced by α via the cycle action from [4]. Assume α is supported on $A \times Y$, where $A \subset X$ is a closed subset of codimension $\geq r$. Then α_ vanishes on $\oplus_{i,j+r>n} H^i(X, W_m \Omega^j)$.*

Proof. We may assume $\alpha = [Z]$, with $Z \subset X \times Y$ an integral closed subscheme of codimension n supported on $A \times Y$. Denote by p_X, p_Y the respective projections from $X \times Y$. It suffices to show for $i \geq 0, j+r > n$, and $b \in H^i(X, W_m \Omega^j)$ that

$$(7.1) \quad p_X^*(b) \cup \mathrm{cl}[Z] = 0 \quad \text{in } H_Z^{i+n}(X \times Y, W_m \Omega_{X \times Y}^{j+n}).$$

Then $\alpha_*(b) = p_{Y*}(p_X^*(b) \cup \mathrm{cl}[Z])$ will also vanish.

We first prove (7.1) for $i = 0$. Denote by $\eta \in X \times Y$ the generic point of Z . Since $W_m \Omega_{X \times Y}^{j+n}$ is Cohen-Macaulay the natural map $H_Z^n(X \times Y, W_m \Omega_{X \times Y}^{j+n}) \rightarrow H_\eta^n(X \times Y, W_m \Omega_{X \times Y}^{j+n})$ is injective. Set $B = \mathcal{O}_{X \times Y, \eta}$ and $C = \mathcal{O}_{X, p_X(\eta)}$; by assumption we have $\dim C \geq r$. Since B is formally smooth over C we find $t_1, \dots, t_r \in C$ and $s_{r+1}, \dots, s_n \in B$ such that $p_X^*(t_1), \dots, p_X^*(t_r), s_{r+1}, \dots, s_n$ form a regular sequence of parameters of B . Hence by [12, II, 3.5] (see also [4, Prop. 2.4.1]), [4, Lem. 3.1.5] and in the notation of [4, 1.11.1] the image of $p_X^*(b) \cup \mathrm{cl}[Z] = \Delta^*(p_X^*(b) \times \mathrm{cl}[Z])$ in $H_\eta^n(X \times Y, W_m \Omega_{X \times Y}^{j+n})$ is up to a sign given by

$$\left[p_X^*(b \cdot d[t_1] \cdots d[t_r]) \cdot d[s_{r+1}] \cdots d[s_n] \right] \\ \left[p_X^*([t_1]), \dots, p_X^*([t_r]), [s_{r+1}], \dots, [s_n] \right].$$

Hence the vanishing follows from $b \cdot d[t_1] \cdots d[t_r] \in W_m \Omega_X^{j+r} = 0$.

For the general case $i \geq 0$, we first observe that the CM property of $W_m \Omega_{X \times Y}^{j+n}$ implies $R\Gamma_Z(W_m \Omega_{X \times Y}^{j+n}) \cong \mathcal{H}_Z^n(W_m \Omega_{X \times Y}^{j+n})[-n]$. Therefore

$$H_Z^{i+n}(X \times Y, W_m \Omega_{X \times Y}^{j+n}) = H^i(X \times Y, \mathcal{H}_Z^n(W_m \Omega_{X \times Y}^{j+n})).$$

Let \mathcal{U} be an open affine cover of X and denote by $\mathcal{U} \times Y$ the open (not necessarily affine) cover of $X \times Y$. We can consider the Čech cohomology with respect to $\mathcal{U} \times Y$ and obtain a natural map

$$(7.2) \quad \check{H}^i(\mathcal{U} \times Y, \mathcal{H}_Z^n(W_m \Omega_{X \times Y}^{j+n})) \rightarrow H^i(X \times Y, \mathcal{H}_Z^n(W_m \Omega_{X \times Y}^{j+n})).$$

Since $\check{H}^i(\mathcal{U}, W_m \Omega_X^j) = H^i(X, W_m \Omega_X^j)$ and pullback and cup product are compatible with restriction to open subsets, we see that $p_X^*(-) \cup \mathrm{cl}[Z] :$

$H^i(X, W_m \Omega_X^j) \rightarrow H_Z^{i+n}(X \times Y, W_m \Omega_{X \times Y}^{j+n})$ naturally factors via (7.2). Therefore the case $i \geq 0$ follows from the case $i = 0$. \square

Theorem 7.2. *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}_k^{n+r}$ be a very general complete intersection of multi-degree d_1, d_2, \dots, d_r such that $d' := \sum_{i=1}^r d_i \leq n + r$ and $n \geq 3$. Let p be a prime, $m \geq 1$, and suppose*

$$(7.3) \quad d_i \geq p^m \cdot \left\lceil \frac{n + r + 1 - d' + d_i}{p^m + 1} \right\rceil$$

for some i . Furthermore, we suppose that p is odd or n is even. Then $p^m | \text{Tor}^{(n-2)}(X)$.

Remark 7.3. Just as for Theorem 6.1, the same result holds for k a field of positive characteristic, but only for $p = \text{char } k$.

Proof. The proof relies on Theorem 7.17, which we prove later in this section.

By Corollary 1.6, we need to find only one smooth complete intersection $X \subset \mathbb{P}_k^{n+r}$ such that $p^m | \text{Tor}^{(n-2)}(X)$.

For a scheme X with locally free sheaf \mathcal{E} and a section $s : \mathcal{O}_X \rightarrow \mathcal{E}$, we let $V(s)$ denote the closed subscheme of X defined by s .

We set $d = d_i$, $a = \left\lceil \frac{n+r+1-d'+d}{p^m+1} \right\rceil$, and $c = d - p^m \cdot a$. Let $\mathcal{O} = W(\bar{\mathbb{F}}_p)$ and $K = \text{Frac}(\mathcal{O})$, we take r, f, g, l , and f_2, \dots, f_r suitably general (we will make this precise) homogeneous polynomials in $\mathcal{O}[X_0, \dots, X_{n+r}]$ of degree $d, d - c, a, 1$, and d_2, \dots, d_r , respectively. We let $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}$ be the closed subscheme of the weighted projective space $\mathbb{P} = \text{Proj } \mathcal{O}[X_0, \dots, X_{n+r}, Y]$, with the X_i having weight 1, and Y having weight a , defined by the homogeneous ideal

$$(7.4) \quad l^c \cdot (Y^{p^m} - f) + p \cdot r, g - p \cdot Y, f_2, \dots, f_r.$$

The generic fiber $X := \mathcal{X}_K$ is isomorphic to the complete intersection of \mathbb{P}_K^{n+r} defined by $l^c \cdot (g^{p^m} - p^{p^m} \cdot f) + p^{p^m+1} \cdot r, f_2, \dots, f_r$. For r, f_2, \dots, f_r general, it is smooth. By replacing \mathcal{O} with its normalization in a suitable finite extension of K and changing notation, we may assume that $\text{Tor}_K^{(n-2)}(X)$ is equal to the geometric torsion order $\text{Tor}^{(n-2)}(X)$.

The special fiber $Y := \mathcal{X}_{\bar{\mathbb{F}}_p}$ is $Y = Y_1 + c \cdot Y_2$. Here, Y_1 is the cyclic p^m cover $Y_1 \rightarrow W$ defined by $f \in H^0(W, \mathcal{O}(a)^{\otimes p^m})$, with $W \subset \mathbb{P}_{\bar{\mathbb{F}}_p}^{n+r}$ the complete intersection defined by g, f_2, \dots, f_r . We will take f, g, f_2, \dots, f_r general enough so that

- (1) W is smooth,
- (2) Y_1 has non-degenerate singularities (see §7.1),
- (3) the assumption (3) of Theorem 7.17 is satisfied for Y_1 .

For (2) we use Proposition 7.5 if $d - c \geq 3$. If $d - c = 2$ hence $p = 2$ then we use Lemma 7.7. For (3) we use the theorem of Illusie about ordinarity of a general complete intersections [16]. Let us check that all other assumptions of Theorem 7.17 are satisfied. (1) is evident, and (2) is equivalent to $(p^m +$

1) $\cdot a - n - r - 1 + d' - d \geq 0$ which follows immediately from the definition of a . Assumption (4) is equivalent to $i \cdot a + a - n - r - 1 + d' - d < 0$, for all $i = 0, \dots, p^m - 1$, which follows from $d' < n + r + 1$; (5) is obvious.

The variety Y_2 is defined by l, g, f_2, \dots, f_r , and only exists if $c \neq 0$. We take l general so that Y_2 does not contain the singular points of Y_1 , $W \cap V(l)$ is smooth, and the p^m cyclic covering of $W \cap V(l)$ corresponding to $f|_{W \cap V(l)}$ has non-degenerate singularities.

Let $r : \tilde{Y}_1 \rightarrow Y_1$ be the resolution of singularities constructed in Proposition 7.8; the map $r : \tilde{Y}_1 \rightarrow Y_1$ is totally CH_0 -trivial. By Proposition 3.3,

$$\text{Tor}^{(n-2)}(X) \cdot \Delta_{\tilde{Y}_1} = \gamma + Z + Z_2,$$

where γ is a cycle with support in $A \times \tilde{Y}_1$ with $\dim A \leq n - 2$, Z has support in $\tilde{Y}_1 \times D$ with D a divisor, and Z_2 has support in $(Y_1 \cap Y_2) \times \tilde{Y}_1$.

In view of Theorem 7.17, we have $\mathbb{Z}/p^m \subset H^0(\tilde{Y}_1, W_m \Omega^{n-1})$. By the work [4] on Hodge-Witt cohomology, we have an action of algebraic correspondences on $H^0(\tilde{Y}_1, W_m \Omega^{n-1})$ (relying on Gros' cycle class [12]). Let us show that Z_2 acts trivially. Note that $T := Y_1 \cap Y_2$ is the p^m cyclic covering of $W \cap V(l)$ corresponding to $f|_{W \cap V(l)}$. An easy computation shows $H^{>0}(Y_1 \cap Y_2, \mathcal{O}) = 0$, hence $H^{>0}(\tilde{T}, \mathcal{O}) = 0$ by Lemma 7.9, where \tilde{T} is the resolution constructed in Proposition 7.8, and $H^{>0}(\tilde{T}, W_m(\mathcal{O})) = 0$. By Ekedahl duality [9], we get $H^{<n-1}(\tilde{T}, W_m \Omega^{n-1}) = 0$. Let \tilde{Z}_2 be a lift of Z_2 to $\tilde{T} \times \tilde{Y}_1$. The action of Z_2 factors as

$$H^0(\tilde{Y}_1, W_m \Omega^{n-1}) \rightarrow H^0(\tilde{T}, W_m \Omega^{n-1}) \xrightarrow{\tilde{Z}_2} H^0(\tilde{Y}_1, W_m \Omega^{n-1}),$$

the first map being the pullback for the map $\tilde{T} \rightarrow \tilde{Y}_1$, thus it is zero.

Lemma 7.1 implies that the action of γ on $H^0(\tilde{Y}_1, W_m \Omega^{n-1})$ vanishes. Therefore

$$\begin{aligned} H^0(\tilde{Y}_1, W_m \Omega^{n-1}) &\xrightarrow{\text{Tor}^{(n-2)}(X)} H^0(\tilde{Y}_1, W_m \Omega^{n-1}) \\ &\xrightarrow{\text{restriction}} H^0(\tilde{Y}_1 \setminus D, W_m \Omega^{n-1}) \end{aligned}$$

is zero. Since the restriction map is injective, we get $p^m | \text{Tor}^{(n-2)}(X)$. \square

Corollary 7.4. *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}_k^{n+r}$ be a very general complete intersection of multi-degree (d_1, \dots, d_r) with $\sum_i d_i = n + r$ and $n \geq 3$. If n is even or d_i is odd then $d_i | \text{Tor}^{(n-2)}(X)$.*

7.1. Let X be a smooth variety over an algebraically closed field k of characteristic p . Suppose that $n := \dim X \geq 2$. Let L be a line bundle on X , and let $s \in H^0(X, L^{\otimes p^m})$. We denote by $\pi : Y \rightarrow X$ the p^m cyclic covering corresponding to s . It is an inseparable morphism and induces an homeomorphism on the underlying topological spaces.

There is a tautological connection $d : L^{\otimes p^m} \rightarrow L^{\otimes p^m} \otimes \Omega_X^1$ which satisfies $d(t^{p^m}) = 0$ for all sections $t \in L$. In particular, we have $d(s) \in H^0(X, L^{\otimes p^m} \otimes \Omega_X^1)$. Note that $Y_{\text{sing}} = \pi^{-1}(V(d(s)))$.

We say that Y has *non-degenerate* singularities if the following conditions hold:

- (1) Y has at most isolated singularities, or equivalently, $\dim(V(d(s))) = 0$ or $V(d(s)) = \emptyset$.
- (2) For all $x \in V(d(s))$, $\text{length}(\mathcal{O}_{V(d(s)),x}) \leq 1$, if p is odd or $p = 2$ and n is even. If $p = 2$ and n is odd then we require $\text{length}(\mathcal{O}_{V(d(s)),x}) \leq 2$ and the blow up $\text{Bl}_x Y$ of x has an exceptional divisor that is a cone over a smooth quadric.

Around a non-degenerate singularity of Y , we can find local coordinates x_1, \dots, x_n of X such that Y is defined by

(7.5)

$$y^{p^m} + x_1^2 + \dots + x_n^2 + f_3 \quad \text{if } p \text{ is odd,}$$

(7.6)

$$y^{p^m} + x_1 x_2 + \dots + x_{n-1} x_n + f_3 \quad \text{if } p = 2 \text{ and } n \text{ is even,}$$

(7.7)

$$y^{p^m} + x_1^2 + x_2 x_3 + \dots + x_{n-1} x_n + b \cdot x_1^3 + f_3 \quad \text{if } p = 2 \text{ and } n \text{ is odd,}$$

where $f_3 \in (x_1, \dots, x_n)^3$, $b \in k^\times$, and f_3 has no x_1^3 term in the last case.

An easy dimension counting argument yields the following proposition (cf. [18, §18]).

Proposition 7.5. *Let $W \subset H^0(X, L^{\otimes p^m})$ be such that for every closed point $x \in X$ the restriction map*

$$W \rightarrow \mathcal{O}_{X,x}/m_x^4 \otimes L^{\otimes p^m}$$

is surjective. For a general section $s \in W$ the corresponding p^m cyclic covering has non-degenerate singularities.

Remark 7.6. If $p \neq 2$ or $\dim X$ even then the following surjectivity is sufficient to conclude the assertion of the proposition:

$$W \rightarrow \mathcal{O}_{X,x}/m_x^3 \otimes L^{\otimes p^m} \quad \text{for every closed point } x \in X.$$

In order to handle the case $d_i = 2 = p, m = 1$, and $n + r + 1 - d' + 2 \leq 3$ in Theorem 7.2 we need the following lemma.

Lemma 7.7. *For a general complete intersection X in \mathbb{P}^{n+r} with $n \geq 2$ and multi-degree (d_1, d_2, \dots, d_r) such that $d_1 \geq 2$, and a general $s \in H^0(\mathbb{P}^{n+r}, \mathcal{O}(2))$, the double covering corresponding to $s|_X$ has non-degenerate singularities.*

Proof. Only the case $p = 2$ and n odd has to be proved. Consider the variety A consisting of points (x, f_1, \dots, f_r, s) where $x \in \mathbb{P}^{n+r}$, (f_1, \dots, f_r, s) are homogeneous of degree $(d_1, \dots, d_r, 2)$, $X = V(f_1) \cap \dots \cap V(f_r)$ is smooth at x , and $d(s)|_X$ is vanishing at x . Those points for which the double covering corresponding to $s|_X$ has non-degenerate singularities at x form an open set B . It is not difficult to show that it is non-empty. Indeed, take $x = [1 : 0 :$

$0 \cdots : 0]$, and (in coordinates x_1, \dots, x_{n+r} around x) $s = 1 + x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n + x_1x_{n+1}$, $f_1 = x_{n+1} + x_1^2$, and $f_i = x_{n+i} + \text{terms of degree } \geq 2$.

Let $V \subset A$ be the open set consisting of points such that $V(f_1) \cap \cdots \cap V(f_r)$ is smooth. Since $B \cap V \neq \emptyset$, we conclude that for a general complete intersection X there is an open non-empty set $U \subset X$ such that for any $x \in U$ the set

$$\{s \in H^0(\mathbb{P}^{n+r}, \mathcal{O}(2)) \mid d(s)|_X(x) = 0 \text{ and } s \text{ does not yield a non-degenerate double covering at } x\}$$

has codimension $\geq n + 1$. Counting dimensions yields the claim. \square

The following proposition has been proved for the case $m = 1$ in [7], and for the general case in [24].

Proposition 7.8. *Suppose Y has non-degenerate singularities. Then by successively blowing up singular points, we can construct a resolution of singularities $r : \tilde{Y} \rightarrow Y$ such that the exceptional divisor is a normal crossings divisor (cf. [18]). Over every singular point $y \in Y$ the fiber $r^{-1}(y)$ is a chain of smooth irreducible divisors, each component of which is either a projective space, a smooth quadric or a projective bundle over a smooth quadric. The intersection of two irreducible components is a smooth quadric or is empty. In particular, since k is algebraically closed, the morphism r is totally CH_0 trivial.*

Proof. We distinguish three cases:

- (1) p is odd,
- (2) $p = 2$, and n is even,
- (3) $p = 2$, and n is odd.

In any case we will only blow up singular points, and over any singular s there will be at most one singular point appearing in the exceptional divisor of the blow up of s .

We may assume that Y has only one singular point. In case (1), note that we have a singularity of the form (7.5). We need $\frac{p^m-1}{2} + 1$ blow ups:

$$\tilde{Y} := Y_{\frac{p^m-1}{2}+1} \rightarrow Y_{\frac{p^m-1}{2}} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 := Y.$$

Around the singularity of Y_i , for $0 \leq i < \frac{p^m-1}{2}$, Y_i is defined by

$$(7.8) \quad y^{p^m-2i} + x_1'^2 + \cdots + x_n'^2 + f_3',$$

where $x_i' = \frac{x_i}{y}$ and $f_3' \in y^i \cdot (x_1', \dots, x_n')^3$. Therefore the exceptional divisor of $Y_{i+1} \rightarrow Y_i$ is the cone C defined by $x_1'^2 + \cdots + x_n'^2$ in the projective space with homogeneous variables y, x_1', \dots, x_n' . For $i = \frac{p^m-1}{2}$, Y_i is also given by (7.8) around the vertex of the exceptional divisor, hence $p^m - 2i = 1$ implies that it is smooth and the exceptional divisor of $Y_{\frac{p^m-1}{2}+1} \rightarrow Y_{\frac{p^m-1}{2}}$ is \mathbb{P}^{n-1} . Denoting by \tilde{E}_i the strict transform in \tilde{Y} of the exceptional divisor of $Y_i \rightarrow Y_{i-1}$, we conclude that \tilde{E}_i is the blow-up of C in its vertex if $i \leq \frac{p^m-1}{2}$,

and $\tilde{E}_{\frac{2^{m-1}}{2}+1} = \mathbb{P}^{n-1}$. Every \tilde{E}_i has only non-empty intersection with \tilde{E}_{i+1} (if $i \leq \frac{2^{m-1}}{2}$) and \tilde{E}_{i-1} (if $i > 1$); the intersection is the smooth quadric given by $x_1'^2 + \cdots + x_n'^2$ in the projective space with homogeneous variables x_1', \dots, x_n' .

For case (2), this case is similar to (1). We need 2^{m-1} blow ups to arrive at \tilde{Y} . Around the singularity of Y_i , for $0 \leq i < 2^{m-1}$, Y_i is defined by

$$(7.9) \quad y^{2^{m-2} \cdot i} + x_1'x_2' + \cdots + x_{n-1}'x_n' + f_3',$$

and the exceptional divisor of $Y_i \rightarrow Y_{i-1}$ is the cone C defined by $x_1'x_2' + \cdots + x_{n-1}'x_n'$ in the projective space P with homogeneous variables y, x_1', \dots, x_n' . The exceptional divisor of $\tilde{Y} := Y_{2^{m-1}} \rightarrow Y_{2^{m-1}-1}$ is the smooth quadric defined by $y^2 + x_1'x_2' + \cdots + x_{n-1}'x_n'$ in P . Again, the intersection of \tilde{E}_i with \tilde{E}_{i-1} is the smooth quadric given by $x_1'x_2' + \cdots + x_{n-1}'x_n'$ in the projective space with homogeneous variables x_1', \dots, x_n' .

For case (3), we need 2^m blow ups to arrive at \tilde{Y} . The case $m = 1$ is easy to check; we will assume $m > 1$. We start with Y and the singularity (7.7). After $2^{m-1} - 1$ blow ups the singularity is of the form

$$b \cdot y^{2^{m-1}+2} + x_1^{[1]2} + x_2'x_3' + \cdots + x_{n-1}'x_n' + b \cdot x_1^{[1]} \cdot y^{2^{m-1}+1} + \text{h.o.t.},$$

where $x_i' = \frac{x_i}{y^{2^{m-1}-1}}$, $x_1^{[1]} = x_1' + y$, and the higher order terms h.o.t. can

be ignored. After 2^{m-2} more blow ups we introduce $x_1^{[2]} = \frac{x_1^{[1]}}{y^{2^{m-2}}} + \sqrt{b} \cdot y$,

after 2^{m-3} more blow ups we introduce $x_1^{[3]} = \frac{x_1^{[2]}}{y^{2^{m-3}}} + \sqrt{\sqrt{b} \cdot b} \cdot y$, etc. The singularity is after $2^{m-1} - 1 + 2^{m-2} + 2^{m-3} + \cdots + 2^{m-i}$ blow ups of the form

$$(7.10) \quad b_i \cdot y^{2^{m-i}+2} + x_1^{[i]2} + x_2'x_3' + \cdots + x_{n-1}'x_n' + b \cdot x_1^{[i]} \cdot y^{2^{m-i}+1} + \text{h.o.t.},$$

where $x_i' = \frac{x_i}{y^{-1+\sum_{j=1}^i 2^{m-j}}}$ and $b_i = b \cdot \sqrt{b_{i-1}}$ with $b_1 = b$. After $2^m - 2$

blow ups we get a singularity (7.10) with $i = m$. After one more blow up the variety becomes smooth, and we need one more blow up to obtain an exceptional divisor with strict normal crossings.

The exceptional divisor E_i of $Y_i \rightarrow Y_{i-1}$ is a cone defined by $x_1^{[i]2} + x_2'x_3' + \cdots + x_{n-1}'x_n'$ in the projective space with homogeneous variables $y, x_1^{[i]}, x_2', \dots, x_n'$, except for the last blow up where it is a projective space. The strict transform \tilde{E}_i is the blow up of the vertex. \square

For p odd or n odd, we get a projective space as exceptional divisor in the last step. Denoting by E the sum over all components of the exceptional divisor of r , we set

$$(7.11) \quad E' := \begin{cases} E + (\text{exc. div. from last step}), & \text{if } p \text{ is odd or } n \text{ is odd,} \\ E & \text{if } p = 2 \text{ and } n \text{ is even.} \end{cases}$$

Thus the exceptional divisor of the last blow up (a projective space) has multiplicity 2 in E' in the first case. If the singularity is of the form (7.5), (7.6), or (7.7), then E' is the restriction of $\text{div}(y)$ to the exceptional divisor of the resolution r .

Lemma 7.9. *The resolution $r : \tilde{Y} \rightarrow Y$ is rational, that is $Rr_*\mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$.*

Proof. We may suppose that Y has only one singularity. We will show that for each $r_i : Y_i \rightarrow Y_{i-1}$, we have $Rr_{i*}\mathcal{O}_{Y_i} = \mathcal{O}_{Y_{i-1}}$. Since Y_{i-1} is normal, it suffices to prove $R^j r_{i*}\mathcal{O}_{Y_i} = 0$. We know that r_i is the blown up of a point and the exceptional divisor D is a cone over a smooth quadric, a smooth quadric, or a projective space, and comes with a given embedding into projective space; we call the corresponding ample line bundle $\mathcal{O}_D(1)$. In any case, $H^{>0}(D, \mathcal{O}(-s \cdot D)) \cong H^{>0}(D, \mathcal{O}(s)) = 0$ for all $s \geq 0$, where $\mathcal{O}_D(s) = \mathcal{O}_D(1)^{\otimes s}$. This implies the claim. \square

Lemma 7.10. *Let E' be as defined in (7.11). For all $i \geq 2$ we have*

$$H^i(E', \mathcal{O}(E')) = 0.$$

Proof. We may suppose that Y has only one singular point. The exceptional divisor is $\sum_{i=1}^s \tilde{E}_i$, and \tilde{E}_i has non-empty intersection only with \tilde{E}_{i+1} and \tilde{E}_{i-1} . Recall that all intersections are smooth quadrics. If $i \neq s$ then \tilde{E}_i is the blow up at the vertex of a cone $C_i \subset \mathbb{P}^n$ over a smooth quadric $Q_i \subset \mathbb{P}^{n-1}$; let $r_i : \tilde{E}_i \rightarrow C_i$ denote the blow up.

For $i = 1, \dots, s-2$, we have $\mathcal{O}_{\tilde{E}_i}(\tilde{E}_i + \tilde{E}_{i+1}) \cong r_i^*\mathcal{O}_{C_i}(-1)$, hence

$$(7.12) \quad \mathcal{O}_{\tilde{E}_i \cap \tilde{E}_{i+1}}(E') \cong \mathcal{O}_{\tilde{E}_i \cap \tilde{E}_{i+1}}.$$

For $i = 2, \dots, s-2$, we obtain $\mathcal{O}_{\tilde{E}_i}(E') \cong \mathcal{O}_{\tilde{E}_i}$.

If p or n is odd then $\mathcal{O}_{\tilde{E}_{s-1}}(\tilde{E}_{s-1} + 2 \cdot \tilde{E}_s) \cong r_{s-1}^*\mathcal{O}_{C_{s-1}}(-1)$ hence $\mathcal{O}_{\tilde{E}_{s-1}}(E') \cong \mathcal{O}_{\tilde{E}_{s-1}}$, and $\mathcal{O}_{\tilde{E}_s}(\tilde{E}_{s-1} + 2 \cdot \tilde{E}_s) \cong \mathcal{O}_{\mathbb{P}^{n-1}}$; thus (7.12) holds for $i = s-1$. If p and n are even then $\mathcal{O}_{\tilde{E}_{s-1}}(\tilde{E}_{s-1} + \tilde{E}_s) \cong r_{s-1}^*\mathcal{O}_{C_{s-1}}(-1)$ hence $\mathcal{O}_{\tilde{E}_{s-1}}(E') \cong \mathcal{O}_{\tilde{E}_{s-1}}$. Moreover, we have $\mathcal{O}_{\tilde{E}_s}(E') \cong \mathcal{O}_{\tilde{E}_s}$. This implies the assertion easily. \square

7.2. Again, we assume that Y has non-degenerate singularities. We denote by $U \subset X$ the complement of the critical points, $Y_{sm} = \pi^{-1}(U)$; we have

$$W_l(\pi)^*W_l\Omega_{U/k}^1 \rightarrow W_l\Omega_{Y_{sm}/k}^1,$$

but there is no Verschiebung on $W_l(\pi)^*W_l\Omega_{U/k}^1$. Therefore we define

$$\text{Im}_V(W_l\Omega_{U/k}^1) \subset W_l\Omega_{Y_{sm}/k}^1$$

inductively on l by

$$\text{Im}_V(W_l\Omega_{U/k}^1) = \text{image}(W_l(\pi)^*W_l\Omega_{U/k}^1 \rightarrow W_l\Omega_{Y_{sm}/k}^1) + V(\text{Im}_V(W_{l-1}\Omega_{U/k}^1)).$$

We have a R, V, F calculus for $\mathrm{Im}_V(W_l\Omega_{U/k}^1)$, that is, morphisms

$$\begin{aligned} R &: \mathrm{Im}_V(W_l\Omega_{U/k}^1) \rightarrow \mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1), \\ V &: \mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1) \rightarrow \mathrm{Im}_V(W_l\Omega_{U/k}^1), \\ F &: \mathrm{Im}_V(W_l\Omega_{U/k}^1) \rightarrow \mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1), \end{aligned}$$

satisfying the relations induced by $W_*\Omega_{Y_{sm}/k}^1$ (see [15]). By abuse of notation, any composition of maps R will be also denoted by R .

We are going to need several statements on $\mathrm{Im}_V(W_l\Omega_{U/k}^1)$ in Theorem 7.17 which we provide in the following.

Lemma 7.11. *The evident map*

$$(7.13) \quad \ker \left(R : W_l(\pi)^* W_l\Omega_{U/k}^1 \rightarrow \pi^* \Omega_U^1 \right) \rightarrow \ker \left(R : \mathrm{Im}_V(W_l\Omega_{U/k}^1) \rightarrow \mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1) \right) / V(\mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1)).$$

is surjective if $l \leq m$.

Proof. The target is the image of $R^{-1}(\ker(\pi^* \Omega_U^1 \rightarrow \Omega_{Y_{sm}}^1)) \subset W_l(\pi)^* W_l\Omega_{U/k}^1$ via the evident map $W_l(\pi)^* W_l\Omega_{U/k}^1 \rightarrow \mathrm{Im}_V(W_l\Omega_{U/k}^1) / V(\mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1))$. Locally, Y_{sm} is defined by $y^{p^m} - f$, for $f \in \mathcal{O}_U$, and $\ker(\pi^* \Omega_U^1 \rightarrow \Omega_{Y_{sm}}^1)$ is generated by $d(f)$. Since $d([f]) \in W_l(\pi)^* W_l\Omega_{U/k}^1$ is a lifting of $d(f)$ whose image vanishes in $\mathrm{Im}_V(W_l\Omega_{U/k}^1)$ (here we use $l \leq m$), the claim follows. \square

Recall the subsheaves $B_n\Omega_{U/k}^1$ of $\Omega_{U/k}^1$, $n = 1, 2, \dots$ (see for example [15, §I.2.2]). We have a short exact sequence

$$W_{l-1}\Omega_{U/k}^1 \xrightarrow{V} \ker \left(R : W_l\Omega_{U/k}^1 \rightarrow \Omega_U^1 \right) \xrightarrow{F^{l-1}} B_{l-1}\Omega_U^1 \rightarrow 0.$$

With the appropriate $W_l(\mathcal{O}_U)$ -module structures this becomes a short exact sequence of $W_l(\mathcal{O}_U)$ -modules. We obtain the following diagram

$$(7.14) \quad \begin{array}{ccc} W_l(\pi)^* \ker \left(R : W_l\Omega_{U/k}^1 \rightarrow \Omega_U^1 \right) / W_l(\pi)^*(V)(W_l(\pi)^* W_{l-1}\Omega_{U/k}^1) & \xrightarrow{\cong} & \pi^* B_{l-1}\Omega_U^1 \\ \downarrow \text{surjective by Lemma 7.11} & & \\ \ker \left(R : \mathrm{Im}_V(W_l\Omega_{U/k}^1) \rightarrow \mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1) \right) / V(\mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1)) & & \\ \downarrow (*) & & \\ \ker \left(R : W_l\Omega_{Y_{sm}/k}^1 \rightarrow \Omega_{Y_{sm}}^1 \right) / V(W_{l-1}\Omega_{Y_{sm}/k}^1) & \xrightarrow{\cong} & B_{l-1}\Omega_{Y_{sm}}^1. \end{array}$$

The induced map

$$(7.15) \quad \pi^* B_{l-1}\Omega_U^1 \rightarrow B_{l-1}\Omega_{Y_{sm}}^1$$

is the natural one, that is, given by $a \otimes \pi^{-1}(\omega) \mapsto \text{Frob}^{l-1}(a) \cdot \pi^{-1}(\omega)$. We would like to show that $(*)$ is injective, which we prove by computing the kernel of (7.15) and showing that it is killed in $\text{Im}_V(W_l \Omega_{U/k}^1)$.

It is convenient to use the isomorphism [15, (I.3.11.4)]

$$(7.16) \quad F^{l-2}d : W_{l-1}(\mathcal{O}_U)/F(W_{l-1}(\mathcal{O}_U)) \xrightarrow{\cong} B_{l-1}\Omega_U^1.$$

The $W_l(\mathcal{O}_U)$ -module structure on the left is via the Frobenius $F : W_l(\mathcal{O}_U) \rightarrow W_{l-1}(\mathcal{O}_U)$. We give $W_{l-1}(\mathcal{O}_{Y_{sm}})/F(W_{l-1}(\mathcal{O}_{Y_{sm}}))$ the analogous $W_l(\mathcal{O}_{Y_{sm}})$ -module structure.

Lemma 7.12. *Suppose Y_{sm} is defined by $y^{p^m} - f$ for $f \in \mathcal{O}_U$ (this is the local picture). The kernel of*

$$W_l(\pi)^*(W_{l-1}(\mathcal{O}_U)/F(W_{l-1}(\mathcal{O}_U))) \rightarrow W_{l-1}(\mathcal{O}_{Y_{sm}})/F(W_{l-1}(\mathcal{O}_{Y_{sm}}))$$

is generated by $V(W_{l-1}(\mathcal{O}_{Y_{sm}})) \otimes W_l(\pi)^{-1}(W_{l-1}(\mathcal{O}_U))$ and elements of the form

$$(7.17) \quad [y^i] \otimes \pi^{-1}(V^j(b)) - [y^{i \% p^{m-1-j}}] \otimes \pi^{-1}(V^j([f^{(i:p^{m-1-j})}] \cdot b)),$$

for all $0 \leq j \leq l-2$, $i \geq p^{m-1-j}$, and $b \in W_{l-1-j}(\mathcal{O}_U)$. Here, $i \% p^{m-1-j}$ means the remainder of i in the division by p^{m-1-j} , and $i = (i : p^{m-1-j}) \cdot p^{m-1-j} + i \% p^{m-1-j}$.

Proof. The kernel contains $V(W_{l-1}(\mathcal{O}_{Y_{sm}})) \otimes W_l(\pi)^{-1}(W_{l-1}(\mathcal{O}_U))$, because $V(a) \otimes \pi^{-1}(b)$ maps to $F(V(a)) \cdot \pi^{-1}(b) = pa \cdot \pi^{-1}(b) = F(V(a \cdot \pi^{-1}(b)))$. Moreover,

$$\begin{aligned} & [y^i] \otimes \pi^{-1}(V^j(b)) - [y^{i \% p^{m-1-j}}] \otimes \pi^{-1}(V^j([f^{(i:p^{m-1-j})}] \cdot b)) \\ & \mapsto [y^{pi}] \cdot V^j(b) - [y^{(i \% p^{m-1-j}) \cdot p}] \cdot V^j([f^{(i:p^{m-1-j})}] \cdot b) \\ & = V^j([y^{p^{1+j} \cdot i}] - [y^{(i \% p^{m-1-j}) \cdot p^{1+j}} \cdot f^{(i:p^{m-1-j})}]) \cdot b = 0. \end{aligned}$$

In order to show that these are all elements in the kernel, we proceed by induction on l . First, we assume $l = 2$. Without loss of generality, we need only consider elements in the kernel that are of the form $\sum_i [y^i] \otimes \pi^{-1}(b_i)$. By étale base change, we may assume that $U = \text{Spec}(k[x_1, \dots, x_n])$ and $x_1 = f$, hence $Y_{sm} = \text{Spec}(k[y, x_2, \dots, x_n])$. By using elements of the form (7.17), we may suppose that $b_i = b_i(x_2, \dots, x_n)$. Since $\sum_i y^{ip} b_i \in k[y^p, x_2^p, \dots, x_n^p]$ implies $b_i \in k[x_2^p, \dots, x_n^p]$, we are done.

Suppose now that $l > 2$. By induction, we need only consider elements in the kernel that are of the form

$$\sum_i [y^i] \otimes \pi^{-1}(V^{l-2}(b_i)),$$

and we may use the same argument as for the $l = 2$ case. \square

Proposition 7.13. *Suppose $l \leq m$. The map*

$$\ker \left(R : \operatorname{Im}_V(W_l \Omega_{U/k}^1) \rightarrow \operatorname{Im}_V(W_1 \Omega_{U/k}^1) \right) / V(\operatorname{Im}_V(W_{l-1} \Omega_{U/k}^1)) \rightarrow \\ \ker \left(R : W_l \Omega_{Y_{sm}/k}^1 \rightarrow \Omega_{Y_{sm}}^1 \right) / V(W_{l-1} \Omega_{Y_{sm}/k}^1)$$

is injective.

Proof. In view of Diagram (7.14) and Lemma 7.12, we need to prove that the following elements vanish in $\operatorname{Im}_V(W_l \Omega_{U/k}^1) / V(\operatorname{Im}_V(W_{l-1} \Omega_{U/k}^1))$,

- (1) $V(a) \cdot dV(b)$ for $a \in W_l(\mathcal{O}_{Y_{sm}})$ and $b \in W_{l-1}(\mathcal{O}_U)$,
- (2) $[y^i] \cdot dV^{j+1}(b) - [y^{i\%p^{m-1-j}}] \cdot dV^{j+1}([f^{(i:p^{m-1-j})}] \cdot b)$ for $b \in W_{l-1-j}(\mathcal{O}_U)$.

For (1), we have

$$V(a) \cdot dV(b) = V(a \cdot d(b)) \in V(\operatorname{Im}_V(W_{l-1} \Omega_{U/k}^1)).$$

For (2), we compute

$$\begin{aligned} [y^i] \cdot dV^{j+1}(b) &= d([y^i] \cdot V^{j+1}(b)) - V^{j+1}(b) \cdot d([y^i]) \\ &= dV^{j+1}([y^{i \cdot p^{1+j}}] \cdot b) - V^{j+1}(b) \cdot d([y^i]) \\ &= dV^{j+1}([y^{(i\%p^{m-1-j}) \cdot p^{1+j}}] \cdot [f^{(i:p^{m-1-j})}] \cdot b) - V^{j+1}(b) \cdot d([y^i]) \\ &= d\left([y^{i\%p^{m-1-j}}] \cdot V^{j+1}([f^{(i:p^{m-1-j})}] \cdot b)\right) - V^{j+1}(b) \cdot d([y^i]) \\ &= V^{j+1}([f^{(i:p^{m-1-j})}] \cdot b) \cdot d([y^{i\%p^{m-1-j}}]) \\ &\quad + [y^{i\%p^{m-1-j}}] \cdot dV^{j+1}([f^{(i:p^{m-1-j})}] \cdot b) - V^{j+1}(b) \cdot d([y^i]), \end{aligned}$$

which together with

$$\begin{aligned} &V^{j+1}([f^{(i:p^{m-1-j})}] \cdot b) \cdot d([y^{i\%p^{m-1-j}}]) - V^{j+1}(b) \cdot d([y^i]) \\ &= V^{j+1}\left(b \cdot \left([f^{(i:p^{m-1-j})}] \cdot F^{j+1}(d([y^{i\%p^{m-1-j}}])) - F^{j+1}(d([y^i]))\right)\right) \\ &= V^{j+1}\left(b \cdot F^{j+1}(d([y^{(i:p^{m-1-j}) \cdot p^{m-1-j}}][y^{i\%p^{m-1-j}}] - [y^i]))\right) = 0 \end{aligned}$$

(note that $F^{j+1}(d([y^{(i:p^{m-1-j}) \cdot p^{m-1-j}}])) = 0$) implies the claim. \square

7.3. We denote by $j : r^{-1}(Y_{sm}) \rightarrow \tilde{Y}$ the open immersion. We will work with the logarithmic de Rham-Witt complex

$$W_l \Omega_{\tilde{Y}/k}^1(\log E) \subset j_* W_l \Omega_{Y_{sm}/k}^1.$$

Locally, when $E = \cup_{i=1}^r V(f_i)$ with $V(f_i)$ smooth, $W_l \Omega_{\tilde{Y}/k}^1(\log E)$ is generated as a $W_l(\mathcal{O}_{\tilde{Y}})$ submodule of $j_* W_l \Omega_{Y_{sm}/k}^1$ by $W_l \Omega_{\tilde{Y}/k}^1$ and $\langle \frac{d[f_i]}{[f_i]} \mid i = 1, \dots, r \rangle$. As for the de Rham complex there is an exact sequence

$$(7.18) \quad 0 \rightarrow W_l \Omega_{\tilde{Y}/k}^1 \rightarrow W_l \Omega_{\tilde{Y}/k}^1(\log E) \rightarrow \bigoplus_{i=0}^r W_l(\mathcal{O}_{V(f_i)}) \rightarrow 0.$$

We have the usual F, V, R calculus for $W_*\Omega_{\tilde{Y}/k}^1(\log E)$.

We define

$$K_l := j_* \text{Im}_V(W_l \Omega_{U/k}^1) \cap W_l \Omega_{\tilde{Y}/k}^1(\log E) \subset j_* W_l \Omega_{Y_{\text{sm}}/k}^1.$$

We have a F, V, R calculus for K_* induced by the one for $\text{Im}_V(W_*\Omega_{U/k}^1)$ and $W_*\Omega_{\tilde{Y}/k}^1(\log E)$. We set $Q_* := W_*\Omega_{\tilde{Y}/k}^1(\log E)/K_*$.

Lemma 7.14. *Suppose that $p \neq 2$ or n is even. Then, for all $l \geq 1$, the following map is surjective:*

$$R : K_l \rightarrow K_1.$$

Proof. The first case is $p \neq 2$. We need to compute K_1 . We may assume that Y has only one singularity as in the proof of Proposition 7.8. Recall that \tilde{Y} is constructed as a sequence of blow ups $\dots \rightarrow Y_i \rightarrow Y_{i-1} \rightarrow \dots \rightarrow Y$. We denote by $r_i : Y_i \rightarrow Y$ the evident composition; we let D_i be the exceptional divisor of r_i , and E_i denotes the exceptional divisor of $Y_i \rightarrow Y_{i-1}$. We would like to understand

$$(7.19) \quad j_{Y_i \setminus D_i, *} \left(\text{image} \left(r_i^* \pi^* \Omega_X^1|_{Y_i \setminus D_i} \rightarrow \Omega_{Y_i \setminus D_i}^1 \right) \right) \cap \Omega_{Y_{i, \text{sm}}}^1(\log D_i|_{Y_{i, \text{sm}}}),$$

in a neighborhood of $E_i \cap Y_{i, \text{sm}}$, where $Y_{i, \text{sm}}$ is the smooth locus of Y_i , and $j_{Y_i \setminus D_i} : Y_i \setminus D_i \rightarrow Y_{i, \text{sm}}$ is the open immersion.

As in the proof of Proposition 7.8, we have coordinates y, x'_1, \dots, x'_n around the singular point of Y_{i-1} , where $x'_j = \frac{x_j}{y^{i-1}}$. We can cover E_i by $n+1$ open sets V_0, V_1, \dots, V_n , where V_0 is a hypersurface in the affine space with coordinates $y, \frac{x'_1}{y}, \dots, \frac{x'_n}{y}$, and V_j is a hypersurface in the affine space with coordinates $\frac{y}{x'_j}, \frac{x'_1}{x'_j}, \dots, \frac{x'_n}{x'_j}$, for $j = 1, \dots, n$. On V_0 we have $E_i \cap V_0 = D_i \cap V_0 = V(y)$. Note that if $i = \frac{p^m-1}{2} + 1$, which is the last blow up, then $E_i \cap V_0$ is empty.

On V_j we have $E_i \cap V_j = V(x'_j)$ and $D_i \cap V_j = V(y)$ if $j = 1, \dots, n$ and $i \notin \{1, \frac{p^m-1}{2} + 1\}$, that is, except for the first and the last blow up. For the first blow up ($i = 1$), we have $E_i \cap V_j = D_i \cap V_j = V(x'_j)$. For the last blow up ($i = \frac{p^m-1}{2} + 1$), we have $E_i \cap V_j = V(x'_j)$ and $D_i \cap V_j = V(\frac{y}{x'_j})$.

We claim that the restriction of (7.19) to V_0 is generated by $\frac{dx_1}{y^i}, \dots, \frac{dx_n}{y^i}$, and the restriction of (7.19) to V_j is generated by $\frac{dx_1}{x_j}, \dots, \frac{dx_j}{x_j}, \dots, \frac{dx_n}{x_j}$. It is obvious that all differential forms are contained in the left hand side of (7.19), and we need to show that they are contained in $\Omega_{Y_{i, \text{sm}}}^1(\log D_i|_{Y_{i, \text{sm}}})$.

Indeed, $\frac{dx_j}{y^i} = \frac{d\left(\frac{x'_j}{y} \cdot y^i\right)}{y^i} = d\left(\frac{x'_j}{y}\right) + i \cdot \frac{x'_j}{y} \cdot \frac{dy}{y}$, and

$$\begin{aligned} \frac{dx_k}{x_j} &= d\left(\frac{x_k}{x_j}\right) + \frac{x_k}{x_j} \cdot \frac{dx_j}{x_j} = d\left(\frac{x'_k}{x'_j}\right) + \frac{x'_k}{x'_j} \cdot \frac{dx_j}{x_j} = \\ &= d\left(\frac{x'_k}{x'_j}\right) + \frac{x'_k}{x'_j} \cdot \left(\frac{dx'_j}{x'_j} + (i-1) \cdot \frac{dy}{y}\right). \end{aligned}$$

In order to show that the given differential forms are generators, we note that the quotient of $\Omega_{Y_{i,\text{sm}}}^1(\log D_i|_{Y_{i,\text{sm}}}) \cap V_j$ by the module generated by these forms is a quotient of a free rank = 1 module. Since the quotient of $\Omega_{Y_{\text{sm}}}^1$ by the image of $\pi^*(\Omega_U^1)$ is free of rank 1, the claim follows.

The case $p = 2$ and n even can be proved in the same way.

In order to prove that $K_l \rightarrow K_1$ is surjective, we may argue by induction on i and only consider a neighborhood of $E_i \cap Y_{i,\text{sm}}$ in $Y_{i,\text{sm}}$. We note that $\frac{dx_j}{y^i}$ can be lifted by $\frac{d[x_j]}{[y^i]} \in K_l(V_0)$, and $\frac{dx_k}{x_j}$ can be lifted by $\frac{d[x_k]}{[x_j]} \in K_l(V_j)$. \square

Remark 7.15. We do not know whether Lemma 7.14 holds if $p = 2$ and n is odd. We can still describe K_1 , but the coordinate changes $x_1^{[1]}, x_1^{[2]}, \dots$ used in the resolution process are incompatible with the multiplicative Teichmüller map and evident liftings do not exist.

7.4. Let us assume that $p \neq 2$ or n is even. In view of the lemma, the map

$$(7.20) \quad \ker(W_* \Omega_{\tilde{Y}/k}^1(\log E) \xrightarrow{R} W_1 \Omega_{\tilde{Y}/k}^1(\log E)) \rightarrow \ker(Q_* \xrightarrow{R} Q_1)$$

is surjective.

As consequence of Proposition 7.13 we obtain the following corollary.

Corollary 7.16. *For all $l \leq m$, the composition*

$$\begin{aligned} \mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{Y}}^{p^{l-1}} &\xrightarrow{dV^{l-2}, \cong} \ker(V : W_{l-1} \Omega_{\tilde{Y}/k}^1 \rightarrow W_l \Omega_{\tilde{Y}/k}^1) \\ &\rightarrow \ker(V : W_{l-1} \Omega_{\tilde{Y}/k}^1(\log E) \rightarrow W_l \Omega_{\tilde{Y}/k}^1(\log E)) \rightarrow \ker(V : Q_{l-1} \rightarrow Q_l) \end{aligned}$$

is surjective on the open set Y_{sm} .

Proof. The first isomorphism follows from [15, Proposition I.3.11]. The second arrow is an isomorphism on Y_{sm} . Set

$$\begin{aligned} A_l &:= \ker\left(R : \text{Im}_V(W_l \Omega_{U/k}^1) \rightarrow \text{Im}_V(W_1 \Omega_{U/k}^1)\right), \\ B_l &:= \ker\left(R : W_l \Omega_{Y_{sm}/k}^1 \rightarrow \Omega_{Y_{sm}}^1\right), \\ C_l &:= \ker\left(R : Q_{l|Y_{sm}} \rightarrow Q_{1|Y_{sm}}\right). \end{aligned}$$

In view of (7.20) we have a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Im}_V(W_{l-1}\Omega_{U/k}^1) & \longrightarrow & W_{l-1}\Omega_{Y_{sm}/k}^1 & \longrightarrow & Q_{l-1|Y_{sm}} \longrightarrow 0 \\ & & \downarrow V & & \downarrow V & & \downarrow V \\ 0 & \longrightarrow & A_l & \longrightarrow & B_l & \longrightarrow & C_l \longrightarrow 0, \end{array}$$

and the snake lemma and Proposition 7.13 imply the assertion. \square

Theorem 7.17. *Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic p . Suppose that p is odd or n is even. Let L be a line bundle on X , and let $s \in H^0(X, L^{\otimes p^m})$ for $m \geq 1$. Suppose that the p^m cyclic covering $\pi : Y \rightarrow X$ corresponding to s has only non-degenerate singularities; let $r : \tilde{Y} \rightarrow Y$ be the resolution from Proposition 7.8. Suppose that*

- (1) $n \geq 3$,
- (2) $H^0(X, L^{\otimes p^m} \otimes K_X) \neq 0$,
- (3) the Frobenius acts bijectively on $H^{n-1}(V(s), \mathcal{O})$,
- (4) $H^n(X, L^{\otimes -j}) = 0$ for all $j = 0, \dots, p^m - 1$,
- (5) $H^{n-1}(X, L^{\otimes -j}) = 0$ for all $j = 0, \dots, p^m$.

Then $W_m(k) \subset H^0(\tilde{Y}, W_m\Omega^{n-1})$.

Proof. We have

$$\mathrm{coker}(\pi^*(\Omega_U^1) \rightarrow \Omega_{Y_{sm}}^1) = \pi^*(L^{-1}),$$

and this identity extends to

$$Q_1 = r^*\pi^*(L^{-1})(E')$$

on \tilde{Y} , with E' as defined in (7.11). If the singularity of Y is of the form (7.5), (7.6), or (7.7), then Q_1 is generated by $\frac{dy}{y}$.

In view of Lemmas 7.9 and 7.10, and conditions (2), (4), and (5), we obtain

$$(7.21) \quad H^{n-1}(\tilde{Y}, Q_1) = 0, \quad H^n(\tilde{Y}, Q_1) \cong H^n(X, L^{\otimes -p^m}) \neq 0.$$

We will work with the short exact sequences

$$(7.22) \quad 0 \rightarrow \ker(R : Q_l \rightarrow Q_1) \rightarrow Q_l \rightarrow Q_1 \rightarrow 0,$$

$$(7.23) \quad Q_{l-1} \xrightarrow{V} \ker(R : Q_l \rightarrow Q_1) \rightarrow T_l \rightarrow 0,$$

where T_l is simply defined to be the cokernel. We claim

$$(7.24) \quad H^{n-1}(\tilde{Y}, T_l) = 0 = H^n(\tilde{Y}, T_l)$$

for all $l \leq m$. The surjectivity of (7.20) yields the surjectivity of the following composition:

$$(7.25) \quad \ker \left(W_l\Omega_{\tilde{Y}/k}^1(\log E) \xrightarrow{R} \Omega_{\tilde{Y}}^1(\log E) \right) / V W_{l-1}\Omega_{\tilde{Y}/k}^1(\log E) \xrightarrow[\cong]{F^{l-1}} B_{l-1}\Omega_{\tilde{Y}}^1 \rightarrow T_l$$

[15, page 575]. Note that

$$\begin{aligned} \ker \left(W_l \Omega_{\tilde{Y}/k}^1 \xrightarrow{R} \Omega_{\tilde{Y}}^1 \right) / V W_{l-1} \Omega_{\tilde{Y}/k}^1 \\ \xrightarrow{\cong} \ker \left(W_l \Omega_{\tilde{Y}/k}^1(\log E) \xrightarrow{R} \Omega_{\tilde{Y}}^1(\log E) \right) / V W_{l-1} \Omega_{\tilde{Y}/k}^1(\log E) \end{aligned}$$

is an isomorphism.

Now we need to find a complex of $W_l(\mathcal{O}_{\tilde{Y}})$ -modules

$$R_1 \rightarrow R_0 \rightarrow \ker(B_{l-1} \Omega_{\tilde{Y}}^1 \rightarrow T_l),$$

such that the following conditions hold:

- $R_0|_{Y_{sm}} \rightarrow \ker(B_{l-1} \Omega_{\tilde{Y}}^1 \rightarrow T_l)|_{Y_{sm}}$ is surjective,
- $H^n(\tilde{Y}, R_1) \rightarrow H^n(\tilde{Y}, R_0)$ is surjective.

It will follow that $H^n(\tilde{Y}, T_l) = 0 = H^{n-1}(\tilde{Y}, T_l)$. Indeed, we have

$$H^n(\tilde{Y}, B_{l-1} \Omega_{\tilde{Y}}^1) = 0 = H^{n-1}(\tilde{Y}, B_{l-1} \Omega_{\tilde{Y}}^1)$$

by induction on l , and using the exact sequence (7.26). The case $l = 2$ follows from assumption (4) and (5), Lemma 7.9, and the short exact sequence (7.27).

We take

$$R_{0,l} := r^* \pi^* B_{l-1} \Omega_X^1, \quad R_{1,l} = \ker(R_{0,l} \rightarrow B_{l-1} \Omega_{\tilde{Y}}^1).$$

Clearly, the image of $r^* \pi^* B_{l-1} \Omega_X^1$ is contained in $\ker(B_{l-1} \Omega_{\tilde{Y}}^1 \rightarrow T_l)$. The surjectivity of $R_0|_{Y_{sm}} \rightarrow \ker(B_{l-1} \Omega_{\tilde{Y}}^1 \rightarrow T_l)|_{Y_{sm}}$ follows from Lemma 7.11 and Diagram (7.14).

We claim that $H^n(\tilde{Y}, R_{1,l}) \rightarrow H^n(\tilde{Y}, R_{0,l})$ is surjective. We will proceed by induction on l . We have an exact sequence of locally free \mathcal{O}_X -modules

$$(7.26) \quad 0 \rightarrow \text{Frob}_*^{l-2} B_1 \Omega_X^1 \rightarrow B_{l-1} \Omega_X^1 \xrightarrow{C} B_{l-2} \Omega_X^1 \rightarrow 0,$$

where C is the Cartier operator. Therefore

$$0 \rightarrow r^* \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1 \rightarrow R_{0,l} \xrightarrow{C} R_{0,l-1} \rightarrow 0$$

is exact. Lemma 7.12 shows that $R_{1,l}|_{Y_{sm}} \xrightarrow{C} R_{1,l-1}|_{Y_{sm}}$ is surjective; note that under the isomorphism $F^{l-2}d$ from (7.16) the Cartier operator corresponds to the restriction. By induction we need to prove that the image of

$$H^n(\tilde{Y}, r^* \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1) \rightarrow H^n(\tilde{Y}, R_{0,l})$$

is contained in the image of $H^n(\tilde{Y}, R_{1,l})$. Rationality of the resolution r , implies

$$\begin{aligned} H^n(\tilde{Y}, r^* \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1) &= H^n(Y, \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1) \\ &= H^n(X, \text{Frob}_*^{l-2}(B_1 \Omega_X^1) \otimes_{\mathcal{O}_X} \pi_* \mathcal{O}_Y). \end{aligned}$$

In view of the exact sequence

$$(7.27) \quad 0 \rightarrow \mathcal{O}_X \xrightarrow{\text{Frob}} \text{Frob}_* \mathcal{O}_X \rightarrow B_1 \Omega_X^1 \rightarrow 0,$$

we obtain a surjective map

$$H^n(X, \bigoplus_{i=p^{m-l+1}}^{p^m-1} L^{-i \cdot p^{l-1}}) \rightarrow H^n(\tilde{Y}, r^* \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1),$$

because

$$\text{Frob}_*^{l-1} \mathcal{O}_X \otimes_{\mathcal{O}_X} \pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{p^m-1} \text{Frob}_*^{l-1}(\text{Frob}^{l-1,*} L^{-i}).$$

For every $p^m > i \geq p^{m-l+1}$, we have two morphisms $\text{Frob}_*^{l-1}(\text{Frob}^{l-1,*}(L^{-i})) \rightarrow \text{Frob}_*^{l-1}(\text{Frob}^{l-1,*}(\pi_* \mathcal{O}_Y))$; the first one is induced by $\text{Frob}_*^{l-1} \text{Frob}^{l-1,*}$ applied to $L^{-i} \subset \pi_* \mathcal{O}_Y$. The second one is induced by Frob_*^{l-1} applied to

$$\begin{aligned} \text{Frob}^{l-1,*}(L^{-i}) &= L^{-i \cdot p^{l-1}} \xrightarrow{s(i \cdot p^{m+1-l})} L^{-(i \% p^{m+1-l}) \cdot p^{l-1}} \\ &= \text{Frob}^{l-1,*}(L^{-(i \% p^{m+1-l})}) \\ &\rightarrow \text{Frob}^{l-1,*}(\pi_* \mathcal{O}_Y), \end{aligned}$$

where the last arrow comes from $L^{-(i \% p^{m+1-l})} \subset \pi_* \mathcal{O}_Y$. Note that after application of $H^n(X, -)$ this map vanishes, because it factors over

$$H^n(X, L^{-(i \% p^{m+1-l}) \cdot p^{l-1}}) = 0.$$

Subtracting the two maps yields a morphism

$$r^* \pi^* \text{Frob}_*^{l-1}(\text{Frob}^{l-1,*}(L^{-i})) \rightarrow (R_{1,l} \cap r^* \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1)$$

which shows that the $H^n(X, L^{-i \cdot p^{l-1}})$ piece of $H^n(\tilde{Y}, r^* \pi^* \text{Frob}_*^{l-2} B_1 \Omega_X^1)$ is contained in the image of $H^n(\tilde{Y}, R_{1,l})$. This proves claim (7.24).

In view of the short exact sequences (7.22), (7.23), Corollary 7.16, vanishing of $H^n(\tilde{Y}, \mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{Y}}^{p^{l-1}})$, and (7.24), we obtain, for all $l \leq m$, a short exact sequence

$$(7.28) \quad 0 \rightarrow H^n(\tilde{Y}, Q_{l-1}) \xrightarrow{V} H^n(\tilde{Y}, Q_l) \xrightarrow{R} H^n(\tilde{Y}, Q_1) \rightarrow 0.$$

This enables us to define

$$\psi_{l-1} : H^n(\tilde{Y}, Q_1) \rightarrow H^n(\tilde{Y}, Q_l), \quad a \mapsto F^{l-1}(R^{-1}(a)).$$

It is evident that $\psi_{l-1} = \psi_1^{l-1}$. In view of (7.21) we have

$$H^n(\tilde{Y}, Q_1) \cong H^n(X, L^{-p^m}).$$

Via this identification, the map ψ_1 is given by

$$H^n(X, L^{-p^m}) \rightarrow H^n(X, L^{-p^{m+1}}) \xrightarrow{\cdot s^{p-1}} H^n(X, L^{-p^m}),$$

where the first arrow is induced by the p -th power map $L^{-p^m} \rightarrow L^{-p^{m+1}}, a \mapsto a^p$. Indeed, denoting $\iota : L^{-p^m} \rightarrow \pi_* r_* Q_1$ the evident map, we have a

commutative diagram

$$\begin{array}{ccccc}
L^{-p^m} & \xrightarrow{(\)^p} & L^{-p^{m+1}} & \xrightarrow{s^{p-1}} & L^{-p^m} \\
\downarrow \wr & & & & \downarrow \wr \\
\pi_* r_* Q_1 & \xrightarrow{\pi_* r_* (F \circ R^{-1})} & \pi_* r_* \left(\frac{Q_1}{\text{image}(B_1 \Omega_{\tilde{Y}}^1)} \right) & \xleftarrow{\pi_* r_* Q_1} & \frac{\pi_* r_* Q_1}{\text{image}(\pi_* r_* B_1 \Omega_{\tilde{Y}}^1)}.
\end{array}$$

Moreover, ψ_1 equals the composition

$$\begin{aligned}
H^n(\tilde{Y}, Q_1) &\xrightarrow{\cong} H^n(X, \pi_* r_* Q_1) \xrightarrow{\pi_* r_* (F \circ R^{-1})} H^n\left(X, \pi_* r_* \left(\frac{Q_1}{\text{image}(B_1 \Omega_{\tilde{Y}}^1)} \right)\right) \\
&\rightarrow H^n\left(\tilde{Y}, Q_1 / \text{image}(B_1 \Omega_{\tilde{Y}}^1)\right) \xrightarrow{\cong} H^n(\tilde{Y}, Q_1),
\end{aligned}$$

where the last morphism is the inverse of the projection $H^n(\tilde{Y}, Q_1) \xrightarrow{\eta} H^n(\tilde{Y}, Q_1 / \text{image}(B_1 \Omega_{\tilde{Y}}^1))$, which is injective, because $H^n(\tilde{Y}, Q_1) \xrightarrow{V} H^n(\tilde{Y}, Q_2)$ factors through η .

In the notation of [3, Definition 1.3.1], we therefore get

$$H_c^n(X \setminus V(s), \mathcal{O})_s \cong \bigcap_{i \geq 1} \text{image}(\psi_1^i).$$

Since $H^{n-1}(X, \mathcal{O}_X) = 0 = H^n(X, \mathcal{O}_X)$, [3, §1.4] implies

$$H_c^n(X \setminus V(s), \mathcal{O})_s \cong H^{n-1}(V(s), \mathcal{O})_s = \bigcap_{i \geq 1} \text{image}(\text{Frob}^i).$$

By using assumption (3), we obtain

$$(7.29) \quad H^n(\tilde{Y}, Q_l) \cong \bigoplus_{i=1}^h W(k)/p^l,$$

where $h = \dim_k H^n(X, L^{-p^m})$. Indeed, since the Frobenius acts bijectively on $H^{n-1}(V(s), \mathcal{O}) \cong H^n(X, L^{-p^m})$, ψ_1 is bijective on $H^n(\tilde{Y}, Q_1)$. In view of (7.28), any lifting of a basis of $H^n(\tilde{Y}, Q_1)$ via the map $R : H^n(\tilde{Y}, Q_l) \rightarrow H^n(\tilde{Y}, Q_1)$ will be a $W(k)/p^l$ -basis of $H^n(\tilde{Y}, Q_l)$.

Finally, let us show that $W_l(k) \subset H^0(\tilde{Y}, W_l \Omega_{\tilde{Y}}^{n-1})$. In view of (7.29), there is a surjective morphism of $W(k)$ -modules

$$H^n(\tilde{Y}, W_l \Omega_{\tilde{Y}/k}^1(\log E)) \rightarrow W(k)/p^l = W_l(k).$$

From the residue short exact sequence (7.18) we obtain a surjective map

$$H^n(\tilde{Y}, W_l \Omega_{\tilde{Y}/k}^1) \rightarrow W_l(k).$$

Ekedahl duality [9] implies

$$R\Gamma(W_l \Omega_{\tilde{Y}}^{n-1}) \xrightarrow{\cong} R\text{Hom}_{W_l(k)}(R\Gamma(W_l \Omega_{\tilde{Y}}^1), W_l(k)[-n]),$$

hence the claim. \square

Remark 7.18. Even for the case $m = 1$ the approach is dual to the one in [18]. With the notation in the proof of Theorem 7.17, we show that the composition

$$H^n(\tilde{Y}, \Omega_{\tilde{Y}}^1) \rightarrow H^n(\tilde{Y}, Q_1) \xrightarrow{\cong} H^n(X, L^{\otimes -p^m})$$

is surjective. For the last isomorphism we use $n \geq 3$, because we need to use Lemma 7.10, where vanishing holds for $i > 1$ only. Since we don't use Lemma 7.14 for this part, the argument also works for $p = 2$ and n odd. Taking duals we obtain an inclusion

$$H^0(X, \omega_X \otimes L^{\otimes p^m}) \subset H^0(\tilde{Y}, \Omega_{\tilde{Y}}^{n-1}).$$

This corresponds to a result about extending $n - 1$ -forms from Y_{sm} to \tilde{Y} in [18] (and [7], [24]).

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